Bootstrapping Long Memory Tests: Some Monte Carlo Results

Anthony Murphy and Marwan Izzeldin
University College Dublin and Cass Business School.

July 2004 - Preliminary

Abstract

We investigate the bootstrapped size and power properties of five long memory tests, including the modified R/S, KPSS and GPH tests. In small samples, the moving block bootstrap controls the empirical size of the tests. However, for these sample sizes, the power of bootstrapped tests against fractionally integrated alternatives is often a good deal less that of asymptotic tests. None of the five tests, either asymptotic or bootstrapped, has high power against a fractionally integrated plus slowly decaying stochastic volatility alternative.

KEY WORDS: Moving block bootstrap; fractional integration.

1 Introduction

Long memory processes, especially fractionally integrated processes, often describe many financial time series, and possibly some macroeconomic ones, rather well. It is important to distinguish long memory processes from more common I(0) and I(1) processes as they imply different long run predictions and responses to shocks (Baillie, 1996). A range of tests for long memory are available. Unfortunately, the evidence is that tests based on asymptotic critical values are often badly sized.

In this paper we report the results of a series of Monte Carlo experiments used to examine the size and power properties of five long memory tests using asymptotic and bootstrapped critical values. We use the moving block bootstrap (MBB) to mimic the dependence in the data. All the test statistics are asymptotically pivotal. This means that, for dependent stationary data satisfying reasonable regularity conditions, bootstrapped critical values should provide a higher order of accuracy than asymptotic critical values. We found this when we used the post-blackened MBB to examine the size and power of the modified R/S statistic (Izzeldin and Murphy, 2000).

The outline of this paper is as follows. We discuss the five tests of long memory in next section. We briefly review the relevant empirical literature on the size and power of these tests in Section 3. We discuss the moving block bootstrap in Section 4 and discuss the Monte Carlo experiments and our findings in Section 5.

2 Tests of Long Memory

We consider five tests of long memory. The five test statistics are Lo's modified rescaled range or R/S statistic (Lo, 1991), the KPSS statistic (Kwiatkowski et al, 1992), the rescaled variance or V/S statistic (Giraitis et. al., 2003), the GPH statistic (Geweke and Porter-Hudak, 1983) and the $\hat{H}$ statistic in Robinson (1995) and Robinson and Henry (1999). The modified R/S, KPSS and V/S statistics for a time series $\{x_t\}$ may be expressed in term of the partial sum of the standardized series $S_T(t) = \sum_{s=1}^{t}(x_s - \bar{x})/(\sqrt{T}\hat{\sigma})$, where $\bar{x} = \frac{1}{T}\sum_{t=1}^{T} x_t$ is the sample mean, $\hat{\sigma}^2$ is a estimate of the long run variance of x_t and T is the sample size. Then, :

$$T^{-\frac{1}{2}}R/S = \max_{0 \le t \le T} S_T(t) - \min_{0 \le t \le T} S_T(t) \tag{1}$$

$$KPSS = \frac{1}{T}\sum_{t=1}^{T} S_T(t)^2 \tag{2}$$

$$V/S = \frac{1}{T}\sum_{t=1}^{T}\left(S_T(t) - \overline{S_T}\right)^2 \tag{3}$$

When $\{x_t\}$ is stationary and under suitable regularity conditions:

$$T^{-\frac{1}{2}}R/S \Rightarrow \max_{0 \le r \le 1} W_1(r) - \min_{0 \le r \le 1} W_1(r) \tag{4}$$

$$KPSS \Rightarrow \int_0^1 W_1(r)^2 dr \tag{5}$$

$$V/S \Rightarrow \int_0^1 W_1(r)^2 dr - \left(\int_0^1 W_1(r) dr\right)^2 \tag{6}$$

where $\Rightarrow$ denotes convergence in distribution, $W(r)$ is a standard Brownian motion process and $W_1(r) = W(r) - rW(1)$ is a standard first order Brownian bridge process.

Giraitis et. al. (2003), *inter alia*, derive the asymptotic distribution of the R/S, KPSS and V/S statistics under short and long memory assumptions. All three test are consistent against fractionally integrated alternatives. In addition, all three tests are asymptotically pivotal, so appropriate bootstrap critical values should outperform asymptotic critical values in smaller samples.

Geweke and Porter-Hudak (1983) showed how to consistently estimate the fractional integration parameter d in an ARFIMA model using a semi-nonparametric, frequency domain procedure and derived its asymptotic distribution. For frequencies near zero, d can be estimated from the least squares regression:

$$ln(I(w_j)) = c - d\, ln(4sin^2(w_j/2)\} + \eta_j, \qquad j = 1, ...n \qquad (7)$$

where $I(w_j)$ is the periodogram of the $\{x_t\}$ series at the n frequencies $w_j = 2\pi j/T$. Often the setting $n = \sqrt{T}$ is chosen. With a proper choice of n, the asymptotic distribution of $\hat{d}$ does not depend on either the order of the ARMA process or on the distribution of the error term in the ARFIMA process $\{x_t\}$. Asymptotically $\hat{d}$ is normally distributed with variance $\pi^2/6$.

Robinson (1995) derives a semi-parametric, frequency domain estimator of the fractional integration parameter d which is closely related to the trimmed Whittle estimator in Kunsch (1987). He refers to it as a Gaussian or local Whittle estimator. The estimator is shown to be consistent and asymptotically normal under relatively weak conditions. Moreover, the asymptotic variance of this estimator is free of unknown parameters. Robinson also shows that it dominates the Geweke and Porter-Hudak (1983) estimator. Robinson and Henry (1999) show that, under weak conditions, these results continue to hold under common forms of conditional heteroscedasticity, of both the long and short memory kind.

3 Previous Monte Carlo Results

In this section we briefly review some of the more recent Monte Carlo results in the literature on testing long memory. Lee and Schmidt (1996) show that the power of the KPSS test against basic fractionally integrated (FI) alternatives in sample sizes ranging from 50 to 500 is comparable to that of the modified R/S test. However, they argue that rather larger sample sizes, such as T = 500 or 1000, are required to distinguish reliably between a long memory process and a short memory process with comparable short-term autocorrelation. Their results show that both tests are sensitive to the choice of lag truncation i.e. the number of covariance terms used to calculate the long run variance $\hat{\sigma}^2$.

Hauser (1997) investigates the size and power properties of the GPH test, the modified R/S test, a semi-parametric frequency domain test due to Robinson (1994) and a test based on the trimmed Whittle likelihood (Kunsch, 1987), *inter alia*. He examined IID, AR(1), MA(1), FI, ARFIMA, GARCH and IGARCH data generation processes but only considered one sample size, namely T = 1000. No single test performed satisfactorily for all of the models considered. He suggests that the R/S statistic is generally robust with the disadvantage of relatively small power. The trimmed Whittle likelihood has high power in general and is robust except for large short run effects.

Teverovsky et. al. (1999) also show that the value of Lo's (1991) modified R/S statistic is sensitive to the choice of the truncation lag used to estimate $\hat{\sigma}^2$. As the truncation lag increases, the test statistic has a strong bias towards accepting the null of no long run dependence, even when the DGP is a basic FI process.

Giraitis et. al. (2003) examined the size and power of the modified R/S, KPSS and V/S statistics using sample sizes of 500 and 1000 using AR(1), FI and long and short memory linear ARCH (Robinson, 1991) DGP's. They find that the V/S statistic achieves a somewhat better balance of size and power than the R/S and KPSS test. They also highlight the sensitivity of the test to the choice of the truncation lag when estimating $\hat{\sigma}^2$.

Robinson and Henry(1999) report an extensive range of Monte Carlo results. They consider IID, ARCH, FI, (nearly integrated) GARCH, EGARCH and long memory linear ARCH models and three sample sizes (T = 64,128 and 256). Their estimator $\hat{H} = \hat{d} - \frac{1}{2}$ appears to perform reasonably well except in the nearly integrated GARCH case.

Hiemstra and Jones (1997) used the original non-parametric bootstrap of Efron (1979), designed for IID observations, to test for long memory in stock returns using the modified R/S statistic. Andersson and Gredenhoff (1998) used the AR-sieve bootstrap in a Monte Carlo experiment looking at the size and power of the modified R/S and GPH tests, as well as a LM test due to Agiaklogou and Newbold (1993), in detecting fractional integration using sample sizes of 750 and 1000 observations. They use four bootstrap resampling procedures. Their basic sieve or residual based bootstrap involves resampling (with replacement) the residuals from an estimated AR model, the maximal order of which is selected using the Bayesian information criterion of Schwartz (1978). They extend this procedure to incorporate ARCH(1) dependence in the residuals. They find that the sieve bootstrap works well in controlling the size of the tests.

4 The Moving Block Bootstrap

The two most common bootstrap procedures for time series are the moving block bootstrap (MBB) and the AR-sieve bootstrap for stationary linear time series (Buhlmann, 2002). Both procedures are easy to implement, at least in principle. However the MBB bootstrap is the more general procedure so we use it in our Monte Carlo experiments. In the MBB, introduced by Kuncsch (1989) and Liu and Singh (1992), the bootstrap sample is obtained by resampling blocks of observations rather than the individual observations. The blocks may overlap. We experimented with the post blackening bootstrap suggested by Davison and Hinkley (1997), which combines the MBB and AR-sieve methods, and obtained no better results than the ones reported below.

Of course, there are some practical and other problems with the MBB (Maddala and Kim, 1998, p. 329-330). For example, the pseudo-time series generated by the moving block method is not stationary even if the original series $\{x_t\}$ is stationary. The choice of block length can be problematic, although the cross-validation procedure in Hall, Horowitz and Jing (1995) and the frequency domain bootstrapping procedures (e.g. Hidalgo, 2003) are worth investigating

In addition, there are few theoretical results on bootstrapping long memory data.

5 The Monte Carlo Experiments

We considered a range of data generation processes (DGP's) in our Monte Carlo experiments. Here we present representative results for five DGP's:

(i) The IID case;

(ii) The AR(1) case;

(iii) The AR(1) with stochastic volatility (SV) case;

(iv) The fractionally integrated (FI) case;

(v) The fractional integrated with stochastic volatility case.

These five cases seem relevant when considering financial data. In the AR(1) case, we set $\rho = 0.5$ which is definitely on the high side for financial data. However if the MBB bootstrap works well with $\rho = 0.5$, it will also work well when the level of autocorrelation is lower. Conditional heteroscedasticity is common in financial data, so we considered a range of GARCH and SV DGP's. The two DGP's generated similar results so we only present the SV results here. The DGP in (iii) is $x_t = (1-0.5L)^{-1}u_t + \exp(h_t/2)\epsilon_t$ with $h_t = 0.95h_{t-1} + \eta_t$ The 0.95 coeffcient on h_{t-1} means that the SV conditional heteoscedasticity is slow to decay. The random errors ϵ_t, u_t and η_t are mean zero, independent normal random variables with variances 1, 0.1 and 0.1 respectively. For the fractionally integrated DGP's, we set the FI parameter d equal to $\frac{1}{3}$, a reasonable value given the range of results in many empirical papers. In the case of (i), (ii) and (iv), we looked at normal and log normal random errors.

Many of the Monte Carlo results summarized in the previous section are based on either rather large or quite small sample sizes. We use four sample sizes - T = 100, 250, 5000 and 1000 - which covers a reasonable range. The Monte Carlo results are based on 1000 replications. A 100 observation "burnin" period is used. The bootstrap results are based on 999 bootstrap replications using the moving block bootstrap with a block length of 10. In general, the results are not sensitive to the choice of block length, as long as it is not too short.

The long run variance $\hat{\sigma}^2$ in the R/S, KPSS and V/S statistics is calculated using $[8\sqrt[4]{T/100}]$ estimated covariance terms - the midpoint of the two settings considered by Lee and Schmidt (1999). We use the standard Newey and West (1987) estimator of $\hat{\sigma}^2$. When calculating the GPH and $\hat{H}$ test statistics, we use $[\sqrt{T}]$ frequency domain terms. All the calculations are carried out in Ox (Doornik, 1998).

6 Results

The Monte Carlo results in Tables 1(a) and 1(b) for the IID case show that the MBB is reasonably successful in controlling the size of all five tests, especially in small samples (T = 100 or 250). This is true for both the normal and non-normal error cases. The empirical and nominal sizes of the asymptotic tests can differ quite a lot, especially for the modified R/S and $\hat{H}$ test when T = 100 or 250. Similar results are obtained in Tables 2(a) and 2(b) using the AR(1) DGP.

We report the results for the AR(1) plus stochastic volatility model in Tables 3(a) and 3(b). The SV component with $h_t = 0.95h_{t-1}+\eta_t$ adds a slowly decaying conditional heteroscedastic error, similar to a GARCH(1,1) error, to the AR(1) error. The sizes of the asymptotic tests can be poor, whereas the nominal and empirical sizes of the bootstrapped tests are reasonably close, even when T = 100.

We report the power of the tests against the fractional integrated FI(d) alternative, with d = 1/3, in Tables 4(a) and 4(b). The power of the tests is higher when the random error is log-normal than when it is normal. The asymptotic tests are generally more powerful than the bootstrapped tests. However, for moderate samples sizes (T $\geq$ 250), the difference in power is generally small, the exception being the $\hat{H}$ test when T = 250. When T $\geq$ 250, the power ranking of the bootstrapped tests appears to be $\hat{H}$, GPH, V/S followed jointly by the KPSS and the modified R/S tests. In small samples, the power of all of the tests, apart from the asymptotic $\hat{H}$ test, is low and the $\hat{H}$ test is not the most powerful one.

Finally. we consider the Monte Carlo results in Tables 5(a) and 5(b) for the FI(d) plus stochastic volatility DGP. Unfortunately, none of the bootstrapped or asymptotic tests has much power given the slowly decaying nature of the SV process. In most cases, there is little difference in power between the bootstrapped and asymptotic tests.

7 Conclusions

To conclude, we find that, with small sample sizes (T = 100 or 250) the moving block bootstrap is helpful in controlling the actual size of the five tests of long memory that we considered. However, for these sample sizes, the power of the bootstrapped test against the basic fractionally integrated alternative is often a good deal less than the power of the corresponding asymptotic test. None of the five tests, either asymptotic or bootstrapped, has high power against the fractionally integrated plus slowly decaying stochastic volatility alternative. The V/S statistic generally performs better in terms of size and than the modified R/S and KPSS statistics. The $\hat{H}$ statistic generally outperforms the GPH statistic and is more powerful than the V/S statistic.

References

[1] Agiakloglou, C. and P. Newbold. "Lagrange Multiplier Tests for Fractional Difference," *Journal of Time Series Analysis*, *15*(3):253–262 (1994).

[2] Andersen, M. and M. Gredenhoff. "Robust Testing for Fractional Integration," *Working Paper Series in Economics and Finance (Stockholm School of Economics.)*, 218 (1998).

[3] Baillie, R. T. "Long Memory Processes and Fractional Integration in Econometrics," *Journal of Econometrics*, *73*:5–59 (1996).

[4] Buhlmann, P. "Bootstrap for Time Series," *Statistical Science*, *17*(1):52–72 (2002).

[5] Davison, A. C. and D. V. Hinkley. *Bootstrap Methods and Their Application*. Cambridge Series in Statistical and Probabilistic Mathematics, Cambridge University Press, 1997.

[6] Doornik, J. A. *GiveWin: An Interface to Empirical Modelling*. London: Timeberlake Consultants Press, 1998.

[7] Efron, B. "Bootstrap Methods: Another Look at the Jackknife," *Annals of Statistics*, *7*:1–26 (1979).

[8] Geweke, J. and S. Porter-Hudak. "The Estimation and Application of Long Memory Time Series Models," *Journal of Time Series Analysis*, *4*(4):221–238 (1983).

[9] Giraitis, L., P. Kokoskva R. Leipus and G. Teyssiere. "Rescaled Variance and Related Tests for Long Memory in Volatility and Levels," *Journal of Econometrics*, *112*:265–294 (2003).

[10] Hall, P., J. Horowitz and B. Jing. "On Blocking Rules for the Bootstrap with Dependent Data," *Biometrika*, *82*(3):561–74 (1995).

[11] Hauser, M. "Semiparametric and Nonparametric Testing for Long Memory: A Monte Carlo Study," *Empirical Economics*, *22*:247–271 (1997).

[12] Hidalgo, J. "An Alternative Bootstrap to Moving Blocks for Time Series Regression Models," *Journal of Econometrics*, *117*:369–399 (2003).

[13] Hiemstra, C. and J. Jones. "Another Look at Long Memory in Common Stock Returns," *Journal of Empirical Finance*, *4* (1997).

[14] Hosking, J. "Fractional Differencing," *Biometrika*, *68*:165–176 (1981).

[15] Izzeldin, M. and A. Murphy. "Bootstrapping the Small Sample Critical Values of the Rescaled Range Statistic.," *The Economic and Social Review*, *31*(4):351–359 (October 2000).

[16] Kunsch, H. R. "The Jackknife and the Bootstrap for General Stationary Observations.," *Annals of Statistics*, *17*:1217–41 (1989).

[17] Kunsh, H. R. "Statistical Aspects of Similar Processes." *Y. Prohorov, Sazanov VV (Eds) Proceedings of the First World Congress of the Bernoulli Society*. VNU Science Press, Utrecht, 1987.

[18] Kwiatkowski, D., P. C. B Phillips P. Scmidt and Y. Shin. "Testing the Null Hypothesis of Stationarity Against the Alternative of a Unit Root: How Sure are We That Economic Time Series Have a Unit Root?," *Journal of Econometrics*, *54*:159–178 (1992).

[19] Lee, D. and P. Schmidt . "On the Power of the KPSS Test of Stationarity Againts Fractionally - Integrated Alternatives," *Journal of Econometrics*, *73*:285–302 (1996).

[20] Liu, R. Y. and K. Singh. "Moving Blocks Jackknife and Bootstrap Capture Weak Dependence." *Exploring the Limits of the Bootstrap* edited by R. Le Page and L. Billard, NewYork: Wiley, 1992.

[21] Lo, Andrew. "Long Term Memory in Stock Market Prices," *Econometrica*, *59*(5):1297–1331 (1991).

[22] Maddala, G. S. and I. M. Kim. *Unit Roots, Cointegration, and Structural Change*. Cambridge University Press, 1998.

[23] Robinson, P. M. "Semiparametric Analysis of Long-Memory Time Series," *The Annals of Statistics*, *22*:515–539 (1994).

[24] Robinson, P. M. "Gaussian Semiparametric Estimation of Long Range Dependence," *The Annals of Statistics*, *23*(5):1630–1661 (1995).

[25] Robinson, P. M. and M. Henry. "Long and Short Memory Conditional Heterskedasticity in Estimating the Memory Parameter of Levels," *Econometric Theory*, *15*:299–336 (1999).

[26] Schwartz, G. "Estimating the Order of a Model.," *The Annals of Statistics*, *2*:461–464 (1978).

[27] Teverovsky, V., M. Taqqu and W. Willinger. "A Critical Look at Lo's Modified R/S Statistic," *Journal of Statistical Planning and Inference*, *80* (1999).

8 Appendix Tables

Table 1(a): Rejection Frequency for IID Model

Sample Size	Test Statistic	Nominal	Normal Random Error 20%	15%	10%	5%	$2\frac{1}{2}$%	1%	Log-Normal (Mean Zero) Random Error 20%	15%	10%	5%	$2\frac{1}{2}$%	1%
T = 100	R/S	Bootstrapped	20.1	13.9	8.6	3.3	1.4	0.4	19.2	13.4	6.4	2.9	0.9	0.2
		Asymptotic	7.1	2.8	0.4	0.1	0.0	0.0	4.3	1.5	0.2	0.0	0.0	0.0
	KPSS	Bootstrapped	21.2	15.8	10.4	5.0	3.3	1.2	22.1	16.3	10.4	5.1	2.6	1.1
		Asymptotic	21.7	15.3	9.4	3.8	1.0	0.0	23.4	16.7	10.0	2.8	1.10	0.0
	V/S	Bootstrapped	20.5	14.7	8.5	3.4	1.4	0.5	20.3	14.2	8.7	4.9	2.0	0.7
		Asymptotic	19.5	12.0	3.8	0.5	0.0	0.0	20.9	12.0	4.6	0.5	0.0	0.0
	GPH	Bootstrapped	16.3	10.6	5.9	3.2	1.4	0.4	18.0	12.1	7.0	2.9	1.0	0.4
		Asymptotic	6.7	5.5	3.7	1.5	0.7	0.2	8.9	6.8	2.9	1.3	0.3	0.3
	$\hat{H}$	Bootstrapped	13.5	8.5	5.5	2.6	1.0	0.1	15.6	10.1	5.3	2.1	0.6	0.1
		Asymptotic	16.3	13.1	10.4	6.9	5.0	3.2	16.9	14.8	10.9	7.1	5.0	2.4
T = 250	R/S	Bootstrapped	21.1	15.0	10.3	5.4	3.4	1.2	19.2	13.5	8.5	4.3	2.2	0.5
		Asymptotic	12.8	8.5	5.0	1.4	0.4	0.0	9.4	5.9	2.6	0.2	0.1	0.0
	KPSS	Bootstrapped	20.5	14.8	10.2	3.8	1.7	0.8	21.5	15.8	10.3	5.3	2.3	0.9
		Asymptotic	20.6	15.5	9.6	3.5	1.2	0.4	21.4	16.0	10.3	4.2	1.4	0.3
	V/S	Bootstrapped	19.5	13.9	9.5	5.3	2.6	1.0	19.9	14.7	9.4	4.7	2.8	1.5
		Asymptotic	18.7	13.1	8.0	3.5	1.1	0.4	20.1	14.0	7.9	3.5	1.2	0.2
	GPH	Bootstrapped	19.9	14.6	10.5	5.1	2.4	1.1	18.8	13.7	9.2	4.8	2.2	0.9
		Asymptotic	10.3	7.9	5.1	2.3	0.9	0.3	8.8	6.9	4.3	2.4	1.2	0.4
	$\hat{H}$	Bootstrapped	19.8	14.6	9.5	4.4	2.0	1.0	18.5	13.2	8.7	4.1	1.5	0.5
		Asymptotic	15.8	13.0	10.7	7.1	4.8	2.9	13.0	11.1	9.0	6.5	4.0	1.8

Notes: The DGP is $x_t = \epsilon_t$ with $\epsilon_t \sim n.i.d.(0,1)$ or, before demeaning, $\ln \epsilon_t \sim n.i.d.(0,1)$.
The Monte Carlo results are based on 1000 replications using a 100 observation "burnin" period.
The bootstrap results are based on 999 bootstrap replications using the moving block bootstrap with a block length of 10.
The long run variance in the R/S, KPSS and V/S statistics is calculated using $[8\sqrt[4]{T/100}]$ estimated covariance terms.
$[\sqrt{T}\ ]$ frequency domain terms are used to calculate the GPH and H test statistics.

Table 1(b): Rejection Frequency for IID Model (Continued)

Sample Size	Test Statistic	Nominal	Normal Random Error						Log-Normal (Mean Zero) Random Error					
			20%	**15%**	**10%**	**5%**	**$2\frac{1}{2}$%**	**1%**	**20%**	**15%**	**10%**	**5%**	**$2\frac{1}{2}$%**	**1%**
T = 500	R/S	Bootstrapped	21.4	15.7	10.2	5.6	2.6	1.3	20.2	14.6	9.8	4.4	2.1	0.9
		Asymptotic	15.7	11.2	7.1	2.6	0.9	0.2	11.9	9.1	4.3	1.4	0.6	0.2
	KPSS	Bootstrapped	19.6	14.8	10.1	5.2	3.2	1.5	21.3	15.6	11.0	5.7	3.0	1.3
		Asymptotic	19.7	14.8	10.2	5.0	2.8	1.1	21.9	15.8	10.5	5.1	2.8	0.7
	V/S	Bootstrapped	20.4	15.9	10.5	5.2	2.9	1.0	21.4	15.1	9.5	4.6	2.6	1.3
		Asymptotic	20.2	16.0	9.3	4.9	1.9	0.5	20.2	14.6	9.5	3.5	1.9	0.6
	GPH	Bootstrapped	19.5	15.3	9.7	4.5	2.2	1.1	18.9	13.7	9.3	5.4	2.8	1.2
		Asymptotic	9.5	6.8	3.9	2.0	0.8	0.3	8.2	7.0	4.9	2.4	0.9	0.6
	$\hat{H}$	Bootstrapped	21.2	16.4	10.9	4.7	1.8	0.7	18.5	13.9	9.1	4.8	2.8	1.0
		Asymptotic	15.3	11.9	9.0	5.0	2.7	1.0	11.9	9.6	7.2	5.2	2.8	1.7
T = 1000	R/S	Bootstrapped	18.7	13.5	9.3	4.2	2.1	0.9	18.4	13.3	9.0	4.0	1.9	0.6
		Asymptotic	14.6	10.6	6.3	3.1	1.3	0.2	13.1	9.4	5.4	1.9	0.8	0.1
	KPSS	Bootstrapped	19.0	15.2	9.8	4.3	2.0	0.7	19.8	14.9	10.7	5.1	2.9	1.4
		Asymptotic	19.1	15.0	9.5	4.0	1.8	0.7	19.4	14.8	10.6	5.1	2.4	1.0
	V/S	Bootstrapped	18.9	14.5	9.2	3.9	1.8	1.1	19.3	13.9	8.8	4.7	2.8	1.5
		Asymptotic	18.2	14.2	9.0	4.0	1.5	0.7	18.8	13.8	8.6	4.3	2.7	0.8
	GPH	Bootstrapped	20.0	15.5	10.4	4.7	2.2	0.9	21.6	15.9	9.9	4.9	2.0	0.8
		Asymptotic	10.2	7.1	4.2	1.6	0.7	0.2	8.9	6.3	4.3	1.4	0.9	0.4
	$\hat{H}$	Bootstrapped	18.7	13.9	8.6	4.7	1.8	0.9	20.7	16.2	11.1	4.7	1.7	0.5
		Asymptotic	11.5	8.9	6.5	3.5	1.9	0.9	13.3	10.7	7.7	3.4	1.8	0.8

Notes: See Table 1(a).

Table 2(a): Rejection Frequency for AR(1) Model with $\rho = 0.5$

Sample Size	Test Statistic	Nominal	Normal Random Error						Log-Normal (Mean Zero) Random Error					
			20%	**15%**	**10%**	**5%**	**$2\frac{1}{2}$%**	**1%**	**20%**	**15%**	**10%**	**5%**	**$2\frac{1}{2}$%**	**1%**
T = 100	R/S	Bootstrapped	17.5	12.6	8.3	3.2	1.1	0.5	17.6	11.8	5.7	2.7	0.6	0.2
		Asymptotic	4.0	1.1	0.2	0.0	0.0	0.0	2.9	0.6	0.2	0.0	0.0	0.0
	KPSS	Bootstrapped	21.4	14.7	10.7	5.5	2.9	0.8	22.4	16.3	10.3	5.1	2.7	1.1
		Asymptotic	26.4	19.6	12.6	5.0	2.0	0.0	27.7	21.2	13.1	4.5	1.6	0.2
	V/S	Bootstrapped	21.4	15.1	9.4	3.4	1.6	0.6	21.5	15.4	9.4	5.0	2.1	0.6
		Asymptotic	25.5	17.2	7.2	1.0	0.0	0.0	26.4	17.4	7.7	1.1	0.1	0.0
	GPH	Bootstrapped	21.1	14.5	7.6	4.0	1.4	0.7	22.1	15.1	9.2	3.6	1.5	0.5
		Asymptotic	17.6	12.3	7.3	4.1	1.9	1.1	18.6	13.7	9.9	4.6	1.6	0.5
	$\hat{H}$	Bootstrapped	18.5	12.0	6.9	3.3	0.7	0.4	19.6	13.5	7.3	2.9	0.8	0.0
		Asymptotic	34.5	30.4	25.2	17.8	13.5	8.3	31.5	27.2	23.7	17.8	13.3	8.6
T = 250	R/S	Bootstrapped	20.7	15.4	10.3	5.6	3.3	1.4	19.0	13.4	8.7	4.7	2.0	0.7
		Asymptotic	13.7	8.7	5.1	1.7	0.4	0.1	11.2	6.9	3.4	0.7	0.2	0.0
	KPSS	Bootstrapped	21.3	15.7	10.0	4.4	1.9	0.6	22.2	16.3	11.1	5.2	2.2	0.8
		Asymptotic	24.5	19.2	12.0	5.4	1.9	0.6	26.1	19.9	13.4	6.1	2.5	0.8
	V/S	Bootstrapped	20.7	14.4	10.1	5.6	3.2	1.2	20.7	15.7	10.6	5.2	2.9	1.6
		Asymptotic	26.4	17.5	11.4	5.5	2.5	0.6	25.4	19.5	12.2	5.3	2.5	0.6
	GPH	Bootstrapped	22.7	16.0	11.1	5.8	2.3	1.3	21.2	15.4	10.6	4.9	2.6	1.0
		Asymptotic	15.3	11.8	8.1	3.9	1.9	0.7	12.5	9.8	7.1	3.4	2.1	0.9
	$\hat{H}$	Bootstrapped	23.5	16.4	10.6	5.3	2.5	1.1	20.8	14.7	9.7	4.0	1.6	0.6
		Asymptotic	25.0	21.5	17.4	12.1	8.3	5.4	21.4	17.3	14.3	9.8	7.7	4.9

Notes: See Table 1(a). The DGP is $x_t = 0.5x_{t-1} + \epsilon_t$ with $\epsilon_t \sim n.i.d.(0,1)$ or, before demeaning, $\ln \epsilon_t \sim n.i.d.(0,1)$.

Table 2(b): Rejection Frequency for AR(1) Model With $\rho = 0.5$ (Continued)

Sample Size	Test Statistic	Nominal	Normal Random Error						Log-Normal (Mean Zero) Random Error					
			20%	**15%**	**10%**	**5%**	**$2\frac{1}{2}$%**	**1%**	**20%**	**15%**	**10%**	**5%**	**$2\frac{1}{2}$%**	**1%**
T = 500	R/S	Bootstrapped	21.1	16.2	11.3	6.2	2.6	1.2	21.3	15.2	9.9	5.0	2.5	0.8
		Asymptotic	19.0	13.7	9.2	3.6	1.7	0.3	16.1	10.9	6.4	2.8	1.0	0.3
	KPSS	Bootstrapped	20.3	15.2	10.5	5.4	3.3	1.4	22.0	16.3	11.3	6.2	3.3	1.5
		Asymptotic	24.4	18.6	12.2	6.8	3.8	1.8	24.7	20.3	14.2	7.5	4.1	1.5
	V/S	Bootstrapped	22.0	16.7	11.9	6.3	3.3	1.2	22.2	17.0	10.7	4.8	3.0	1.6
		Asymptotic	26.0	20.6	14.4	7.1	3.7	1.1	26.4	20.4	13.4	6.3	3.2	1.1
	GPH	Bootstrapped	20.9	16.0	10.3	5.2	2.3	0.9	19.9	15.3	10.6	6.2	3.1	1.3
		Asymptotic	11.3	9.0	5.5	2.5	1.2	0.4	10.6	8.4	6.1	3.2	1.7	0.7
	$\hat{H}$	Bootstrapped	21.9	18.0	12.1	5.8	2.0	0.8	20.3	15.1	10.2	5.1	2.9	1.3
		Asymptotic	19.6	16.2	12.1	7.9	4.9	2.1	15.6	12.6	9.9	6.3	4.8	2.4
T = 1000	R/S	Bootstrapped	20.2	14.2	9.9	4.6	2.5	1.1	19.4	14.0	9.5	4.5	2.1	0.8
		Asymptotic	20.3	13.9	9.6	4.4	2.1	0.6	18.2	12.5	8.8	3.8	1.3	0.5
	KPSS	Bootstrapped	19.5	15.3	10.4	5.1	1.9	0.8	20.0	15.4	11.1	6.0	3.1	1.3
		Asymptotic	23.3	17.7	13.1	6.6	2.9	1.1	23.6	18.6	12.7	6.7	3.5	1.6
	V/S	Bootstrapped	20.3	15.1	10.0	4.4	2.1	1.1	20.5	14.4	9.7	4.9	3.2	1.5
		Asymptotic	25.8	18.5	13.3	5.7	3.0	1.1	24.3	19.5	12.5	6.7	3.5	2.0
	GPH	Bootstrapped	20.9	16.1	11.0	5.8	2.7	1.3	21.5	16.4	10.6	5.4	2.6	0.7
		Asymptotic	11.3	8.5	6.0	2.3	1.2	0.2	10.6	7.6	5.2	2.2	0.9	0.5
	$\hat{H}$	Bootstrapped	19.6	14.2	9.7	4.9	2.3	1.0	21.8	16.7	11.9	5.3	2.0	0.7
		Asymptotic	13.5	11.0	7.8	4.4	2.6	1.3	15.8	12.7	9.9	5.0	2.2	1.0

Notes: See Table 2(a).

Table 3(a): Rejection Frequency for AR(1) plus SV Model with $\rho = 0.5, \gamma = 0.95$								
Sample Size	**Test Statistic**	Nominal	**Normal Random Error**					
			20%	**15%**	**10%**	**5%**	**$2\frac{1}{2}$%**	**1%**
T = 100	R/S	Bootstrapped	19.0	12.9	8.2	4.4	2.1	0.7
		Asymptotic	7.7	4.0	1.7	0.2	0.0	0.0
	KPSS	Bootstrapped	21.8	15.9	11.6	5.3	2.4	1.2
		Asymptotic	23.2	17.5	10.9	3.9	1.1	0.1
	V/S	Bootstrapped	22.6	16.7	10.1	5.2	2.5	1.1
		Asymptotic	23.3	14.9	6.5	0.9	0.0	0.0
	GPH	Bootstrapped	19.5	13.7	7.7	2.6	1.4	0.4
		Asymptotic	11.7	8.0	4.6	2.6	1.1	0.6
	$\hat{H}$	Bootstrapped	17.3	11.7	7.0	2.0	0.6	0.2
		Asymptotic	21.8	18.5	15.5	11.0	7.7	5.0
T = 250	R/S	Bootstrapped	21.5	16.8	10.3	4.3	2.2	0.9
		Asymptotic	14.1	7.8	3.7	1.4	0.4	0.0
	KPSS	Bootstrapped	20.5	14.9	10.2	5.2	2.5	0.9
		Asymptotic	19.8	15.5	10.2	4.9	1.7	0.3
	V/S	Bootstrapped	22.4	15.8	9.8	4.8	1.6	0.5
		Asymptotic	21.3	15.2	9.0	3.4	1.0	0.4
	GPH	Bootstrapped	23.2	18.0	11.4	5.2	2.8	0.8
		Asymptotic	11.4	8.7	5.6	2.6	1.4	0.6
	$\hat{H}$	Bootstrapped	22.7	16.9	10.5	4.8	2.3	1.0
		Asymptotic	19.9	15.9	11.4	7.8	5.3	3.6

Notes: See Table 1(a). The DGP is $x_t = (1 - 0.5L)^{-1}u_t + \exp(h_t/2)\epsilon_t$ with $h_t = 0.95h_{t-1} + \eta_t$. ϵ_t, u_t and η_t are mean zero, independent normal random variables with variances 1, 0.1 and 0.1 respectively.

Table 3(b): Rejection Frequency for AR(1) plus SV Model (Continued)

Sample Size	**Test Statistic**	Nominal	**Normal Random Error**					
			20%	**15%**	**10%**	**5%**	**$2\frac{1}{2}$%**	**1%**
T = 500	R/S	Bootstrapped	22.6	17.1	11.2	5.2	3.2	1.1
		Asymptotic	16.9	11.1	6.5	2.7	0.9	0.1
	KPSS	Bootstrapped	20.1	15.0	10.0	4.8	2.2	1.1
		Asymptotic	20.3	15.6	10.7	5.2	2.4	0.9
	V/S	Bootstrapped	20.5	15.8	10.4	5.5	2.6	1.0
		Asymptotic	20.7	15.8	9.9	4.6	1.9	0.6
	GPH	Bootstrapped	22.8	18.8	12.7	6.4	3.7	1.7
		Asymptotic	11.8	9.7	6.3	3.2	1.7	0.3
	$\hat{H}$	Bootstrapped	23.9	19.0	11.6	6.6	3.6	1.9
		Asymptotic	16.8	13.4	10.3	7.2	4.3	2.5
T = 1000	R/S	Bootstrapped	23.2	17.7	11.6	5.5	2.4	0.7
		Asymptotic	19.1	14.4	8.8	3.1	1.2	0.6
	KPSS	Bootstrapped	19.2	14.0	10.1	4.4	2.1	0.9
		Asymptotic	19.8	14.6	9.7	4.1	2.3	0.8
	V/S	Bootstrapped	20.6	16.3	11.3	5.8	2.3	0.8
		Asymptotic	21.7	16.4	10.7	5.5	2.1	0.6
	GPH	Bootstrapped	24.5	18.3	13.0	6.8	4.2	1.8
		Asymptotic	11.8	8.5	6.7	3.2	1.7	0.7
	$\hat{H}$	Bootstrapped	22.8	18.9	13.0	6.0	3.5	2.4
		Asymptotic	15.9	12.8	9.0	5.3	3.3	2.3

Notes: See Table 3(a).

Table 4(a): Rejection Frequency for FI Model with d = $\frac{1}{3}$

Sample Size	Test Statistic	Nominal	Normal Random Error						Log-Normal (Mean Zero) Random Error					
			20%	**15%**	**10%**	**5%**	**$2\frac{1}{2}$%**	**1%**	**20%**	**15%**	**10%**	**5%**	**$2\frac{1}{2}$%**	**1%**
T = 100	R/S	Bootstrapped	27.9	19.9	11.4	5.0	2.5	1.1	30.2	20.5	12.7	6.2	1.8	0.4
		Asymptotic	7.1	3.0	0.5	0.0	0.0	0.0	7.3	2.1	0.2	0.0	0.0	0.0
	KPSS	Bootstrapped	38.6	33.3	27.0	16.7	10.7	6.0	41.8	35.7	28.9	17.9	11.2	6.1
		Asymptotic	46.9	39.1	32.0	19.0	10.3	2.3	49.6	43.1	34.0	20.5	10.7	1.9
	V/S	Bootstrapped	40.5	33.0	24.0	12.9	6.7	2.3	44.0	36.3	26.2	14.2	7.2	2.8
		Asymptotic	48.1	39.5	24.3	6.7	0.7	0.0	53.3	42.2	26.1	6.4	0.5	0.0
	GPH	Bootstrapped	41.0	31.0	20.0	8.9	4.2	1.6	40.6	30.7	18.7	8.3	3.7	1.3
		Asymptotic	47.8	40.4	31.1	20.3	11.3	5.5	46.7	39.9	30.9	20.5	12.7	6.7
	$\hat{H}$	Bootstrapped	42.6	30.6	18.2	8.3	3.9	1.1	43.0	29.2	17.2	7.5	3.2	0.9
		Asymptotic	67.6	63.1	58.6	51.0	44.6	36.0	69.9	66.4	62.8	51.6	44.6	35.1
T = 250	R/S	Bootstrapped	53.4	47.4	39.6	27.6	18.2	11.4	55.1	47.8	40.1	28.4	19.7	11.5
		Asymptotic	48.5	41.9	30.8	16.3	7.9	2.4	47.2	39.8	31.0	16.6	8.0	2.8
	KPSS	Bootstrapped	52.2	44.8	36.8	25.7	18.9	12.5	53.9	47.1	37.9	27.4	20.9	13.4
		Asymptotic	60.9	53.4	43.8	30.7	23.1	14.4	58.1	52.4	44.1	32.8	24.2	16.5
	V/S	Bootstrapped	57.2	51.2	41.7	30.3	22.8	15.4	62.2	56.2	47.1	33.6	23.5	15.9
		Asymptotic	66.2	59.4	50.3	37.0	25.0	14.8	58.1	52.4	44.1	32.8	24.2	16.5
	GPH	Bootstrapped	66.1	58.8	48.4	32.4	21.1	11.9	67.4	60.3	51.5	37.7	27.2	15.9
		Asymptotic	65.4	58.3	50.8	39.4	27.1	17.5	67.5	61.3	47.9	30.5	18.6	9.8
	$\hat{H}$	Bootstrapped	75.8	68.4	56.4	37.6	24.8	11.8	77.8	69.5	57.5	35.3	20.5	8.7
		Asymptotic	83.0	79.9	75.2	68.2	61.1	51.1	84.1	81.5	77.9	69.9	61.3	52.6

Notes: See Table 1(a). The DGP is $x_t = (1-L)^{-\frac{1}{3}}\epsilon_t$ with $\epsilon_t \sim n.i.d.(0,1)$ or, before demeaning, $\ln \epsilon_t \sim n.i.d.(0,1)$.

Table 4(b): Rejection Frequency for FI Model with d = $\frac{1}{3}$ (Continued)

Sample	Test		Normal Random Error						Log-Normal (Mean Zero) Random Error					
Size	**Statistic**	Nominal	**20%**	**15%**	**10%**	**5%**	**$2\frac{1}{2}$%**	**1%**	**20%**	**15%**	**10%**	**5%**	**$2\frac{1}{2}$%**	**1%**
T = 500	R/S	Bootstrapped	72.1	67.0	59.9	49.7	40.0	29.2	74.6	69.0	61.7	50.0	40.3	32.1
		Asymptotic	73.1	67.9	60.3	49.3	38.5	25.1	73.9	67.6	60.4	46.8	37.7	26.7
	KPSS	Bootstrapped	66.6	59.9	51.3	37.9	30.9	23.6	66.5	60.5	52.4	40.3	31.2	24.1
		Asymptotic	74.6	68.6	59.5	46.2	36.6	27.8	74.5	68.6	60.5	47.1	38.1	28.0
	V/S	Bootstrapped	73.7	68.1	60.1	49.9	40.7	31.0	74.6	68.7	61.2	49.5	40.7	31.8
		Asymptotic	82.0	76.0	70.4	58.1	47.8	37.0	82.6	76.9	69.4	56.6	47.3	36.1
	GPH	Bootstrapped	81.5	77.0	69.9	55.3	43.4	31.0	84.2	78.6	68.9	56.0	42.8	28.6
		Asymptotic	77.1	72.6	64.4	51.2	40.0	27.5	77.9	71.6	64.7	51.8	39.3	26.1
	$\hat{H}$	Bootstrapped	89.0	85.2	80.3	67.8	55.9	40.5	90.9	86.9	81.0	68.5	53.9	39.1
		Asymptotic	89.8	87.7	84.4	78.2	71.6	63.0	91.7	89.7	85.0	79.1	70.7	62.3
T = 1000	R/S	Bootstrapped	87.0	82.9	77.1	67.5	60.5	50.9	88.6	86.0	80.0	71.2	63.3	53.8
		Asymptotic	90.0	86.6	81.7	72.6	64.1	55.2	90.8	87.7	83.2	74.0	66.7	55.7
	KPSS	Bootstrapped	78.5	72.7	65.4	53.5	44.6	34.7	79.8	72.6	64.9	53.5	43.3	34.3
		Asymptotic	85.3	80.1	72.8	61.6	53.0	42.7	86.4	81.1	73.3	61.6	52.5	40.9
	V/S	Bootstrapped	85.2	81.0	75.5	66.1	56.6	46.3	86.3	83.3	78.0	68.3	59.6	51.0
		Asymptotic	91.4	87.8	82.7	75.1	66.7	55.9	92.0	89.1	84.0	76.5	68.8	58.8
	GPH	Bootstrapped	92.9	90.0	85.8	78.1	69.1	55.4	93.4	91.2	87.0	79.4	70.1	56.4
		Asymptotic	88.4	85.1	80.6	72.6	61.7	48.3	89.9	87.2	81.8	72.9	63.2	48.1
	$\hat{H}$	Bootstrapped	97.0	95.9	93.8	89.8	83.1	75.0	97.7	96.4	94.7	89.4	83.7	74.0
		Asymptotic	96.4	95.3	94.0	91.2	87.9	81.7	97.2	96.5	94.7	92.2	87.8	81.7

Notes: See Table 4(a).

Table 5(a): Rejection Frequency for FI plus SV Model								
Sample Size	**Test Statistic**	Nominal	**Nominal Size of Test** 20%	15%	10%	5%	$2\frac{1}{2}$%	1%
T = 100	R/S	Bootstrapped	20.3	15.1	9.6	5.7	2.2	0.6
		Asymptotic	9.7	4.9	1.8	0.2	0.0	0.0
	KPSS	Bootstrapped	25.0	19.7	13.6	7.7	4.3	1.6
		Asymptotic	27.0	20.2	13.6	6.1	1.8	0.3
	V/S	Bootstrapped	24.9	19.4	12.5	6.8	3.2	1.4
		Asymptotic	26.2	17.2	8.5	1.7	0.0	0.0
	GPH	Bootstrapped	22.3	15.5	8.6	3.5	2.1	0.5
		Asymptotic	14.1	10.1	6.8	2.8	1.6	0.7
	$\hat{H}$	Bootstrapped	19.7	13.4	7.9	2.8	0.8	0.4
		Asymptotic	24.6	21.7	18.5	14.3	9.8	6.1
T = 250	R/S	Bootstrapped	30.3	23.5	16.5	8.3	4.7	1.8
		Asymptotic	21.7	14.0	7.7	2.7	1.0	0.1
	KPSS	Bootstrapped	30.4	25.2	17.9	10.5	5.7	2.4
		Asymptotic	31.2	25.6	17.9	10.2	4.4	1.5
	V/S	Bootstrapped	31.4	25.9	19.0	9.5	5.2	2.2
		Asymptotic	32.6	25.3	17.4	7.7	3.7	0.7
	GPH	Bootstrapped	30.8	22.5	16.2	8.4	4.3	2.0
		Asymptotic	18.5	14.1	9.4	4.9	2.7	1.4
	$\hat{H}$	Bootstrapped	32.4	25.0	17.3	8.3	3.8	1.3
		Asymptotic	28.9	24.8	21.3	14.8	10.2	5.9

Notes: See Table1(a). The DGP is $x_t = (1-L)^{-\frac{1}{3}}u_t + \exp(h_t/2)\epsilon_t$ with $h_t = 0.95h_{t-1} + \eta_t$. ϵ_t, u_t and η_t are mean zero, independent normal random variables with variances 1, 0.1 and 0.1 respectively.

Table 5(b): Rejection Frequency for FI plus SV Model (Continued)								
Sample Size	**Test Statistic**	Nominal	**Normal Random Error**					
			20%	**15%**	**10%**	**5%**	**$2\frac{1}{2}$%**	**1%**
T = 500	R/S	Bootstrapped	38.4	32.2	22.9	14.0	8.1	4.3
		Asymptotic	32.4	25.3	16.7	8.2	3.9	1.9
	KPSS	Bootstrapped	33.6	28.3	21.3	13.0	8.2	4.0
		Asymptotic	35.1	28.7	21.7	13.8	8.2	3.9
	V/S	Bootstrapped	35.8	28.7	22.7	15.2	9.2	4.1
		Asymptotic	36.7	29.6	23.2	14.4	7.8	3.7
	GPH	Bootstrapped	36.4	29.3	22.5	13.0	7.5	3.7
		Asymptotic	21.8	18.2	12.4	7.3	4.0	1.7
	$\hat{H}$	Bootstrapped	38.8	31.8	24.0	15.5	8.7	3.8
		Asymptotic	31.4	27.4	23.4	17.1	12.0	7.1
T = 1000	R/S	Bootstrapped	49.0	41.4	33.6	23.7	15.8	9.3
		Asymptotic	45.4	37.6	29.9	19.5	12.4	6.5
	KPSS	Bootstrapped	43.9	36.6	28.3	18.5	12.7	7.2
		Asymptotic	44.5	38.2	29.5	18.5	13.6	6.7
	V/S	Bootstrapped	46.4	40.9	32.3	21.1	15.5	9.5
		Asymptotic	48.5	42.5	33.0	21.3	14.7	9.1
	GPH	Bootstrapped	46.3	38.5	30.2	18.6	12.3	7.0
		Asymptotic	29.5	23.9	17.3	11.0	6.9	3.4
	$\hat{H}$	Bootstrapped	50.7	44.5	35.0	23.8	15.8	9.1
		Asymptotic	40.9	36.3	30.6	22.6	16.1	10.5

Notes: See Table 5(a).