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Affine-structure Models and the Pricing of Energy Commodity Derivatives

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Abstract

We consider a seasonal mean-reverting model for energy commodity prices with jumps and Heston-type stochastic volatility, as well as three nested models for comparison. By exploiting the affine form of the log-spot models, we develop a general valuation framework for futures and discrete arithmetic Asian options. We investigate five major petroleum commodities from the European market (Brent crude oil, gasoil) and US market (light sweet crude oil, gasoline, heating oil) and analyze the effects of the competing fitted stochastic spot models in futures pricing, Asian options pricing and hedging. We find evidence that price jumps and stochastic volatility are important features of the petroleum price dynamics.

Keywords: energy prices, affine models, futures, arithmetic Asian options, control variate Monte Carlo

1 Introduction

Understanding the stochastic process governing the energy price is essential, owing to the indispensable role of hydrocarbons in the world economy and the response of macroeconomic aggregates to oil price shocks. Concerns about the security of energy supply and the influence of geopolitical events and global economic activity on petroleum prices create the need for reliable and efficient tools to price energy-related securities and projects.

Energy commodities are predominantly different from conventional financial assets such as equity and fixed-income securities and, due to intricate price formation mechanisms, traditional modelling techniques are not directly applicable. For example, seasonal effects arise naturally from periodic supply and demand patterns and have been successfully modelled by several authors including Routledge et al. (2000) and Borovkova and Geman (2006). In addition, commodity prices mean-revert to the marginal cost of production; relevant theoretical arguments and empirical evidence have been put forward by Bessembinder et al. (1995), Schwartz and Smith (2000) and Casassus and Collin-Dufresne (2005), among others. In effect, prices may temporarily be high or low, but will tend toward an equilibrium level. Furthermore, temporary supply and demand imbalances, changes in market expectations, or even unanticipated macroeconomic developments may cause sudden jumps in energy prices (Hilliard and Reis, 1998, Clewlow and Strickland, 2000). Due to construction lags on the supply side, even a relatively small change in demand can, at times, cause immediate market movements of large magnitude. Yet, jumps in returns are transient and a more persistent component may be required. In fact, compared to other markets, energy price volatility is both relatively higher and more variable over time (see, e.g., Pindyck, 2004). Trolle and Schwartz (2009) develop a stochastic volatility model for crude oil and highlight

\footnote{For example, heating oil prices experience an upward pressure during winter when storage capacity may not be able to smooth out seasonal demand shocks, particularly in the peak demand season. Instead, lower prices are anticipated during the summer inventory build-up period. On the other hand, gasoline prices typically trade at a discount during winter, as demand falls from the peak levels of the summer driving season. Crude oil demand is derived from the demand for its refined products and, as such, seasonal patterns are less evident.}
its importance in commodity derivatives pricing. Larsson and Nossman (2011) find that, in addition to stochastic volatility, jumps are essential to capture the time series properties of oil prices. Accounting for both jumps and stochastic volatility provides a reasonable characterization of energy commodity prices in that it explains the skewness and fat-tail feature of commodity return distributions. Furthermore, it gives rise to realistic implied volatility patterns for short-term options without also affecting long-term smiles.

The previous discussion encourages the empirical testing of different spot model specifications for further use as inputs in derivatives pricing and risk management, energy investment evaluation, asset allocation and planning. The aim of this paper is to conduct a comprehensive analysis of stochastic dynamic modelling of European and US petroleum commodity prices and enrich existing literature with some new insights in several applications such as futures pricing, options pricing and hedging. In terms of scope our work shares similarities with recent contributions in the finance literature focused on other markets like, for example, the equity index (see Kaeck and Alexander, 2013), freight (see Nomikos et al., 2013) and real estate (see Fabozzi et al., 2012) markets. More specifically, we consider a seasonal mean-reverting spot price model with compound Poisson jumps and Heston-type stochastic volatility, as well as three nested models for comparison: a diffusion with stochastic volatility, a jump diffusion with constant volatility, and the classical Schwartz (1997) model with constant volatility. The Heston (1993) volatility model is both mean-reverting and positive, accounts for volatility clustering, dependence in increments and realistic implied volatility patterns. Furthermore, it allows a simple description of the correlation between the driving noises in the returns and volatility processes; in particular, positive correlation is interpreted in terms of the so-called inverse leverage effect, i.e., the observation that in commodity markets large upward price moves are associated with high volatility due to negative relationship between inventory and prices (e.g., see Pindyck, 2004, Geman, 2005).

The contribution of our paper is three-fold. Firstly, we derive the bivariate characteristic function of the suggested jump diffusion model with stochastic volatility by solving a system
of Riccati equations. We then use our results to obtain expressions for the theoretical futures prices, tackling previously noted (see [Benth, 2011]) mathematical challenges originated by the mean-reverting term in the price dynamics in the Heston stochastic volatility framework and rendering this useful for calibration purposes. Secondly, we fit the models to spot and futures prices of Brent crude oil and gasoil from the European market and light sweet crude oil, gasoline and heating oil from the US market. This is the first study to systematically apply a selection of stochastic models in the particular markets. We find that ignoring jumps and/or stochastic volatility leads to a less realistic description of the true data-generating process (DGP). The flexibility of the proposed general model specification is also confirmed by its ability to accurately fit the observed futures curves in the different markets. The final line of research that we contribute to in this paper relates to average (Asian) options, whose terminal payoff depends on the average level of an energy price during a pre-specified time window (e.g., see [Zhang, 1995]). These options are very popular in the energy commodity markets as a means of managing price exposure and potential impact on transactions, due to the time elapsed until a tanker vessel completes its route from the production site or refinery to its destination. Using our characteristic function results we obtain closed-form solutions for geometric average options, which we then use to implement efficient Monte Carlo simulation with control variates to price the more prevalent discrete arithmetic average options. This way we present a unified pricing framework under the four models considered, while extending earlier contributions by [Kemna and Vorst, 1990] and [Fusai and Meucci, 2008] based, respectively, on Gaussian and Lévy log-spot models to members from the more general affine class. The particular technique provides fast and accurate simulation outcome, which allows us to study the implications of the assumed spot price dynamics on the Asian option prices. In addition, we set up delta hedge portfolios for the Asian option and investigate their performance when the underlying spot price process is driven by the adopted models. We find that, for different possible representations of the true DGP, a hedge based on the jump diffusion model with stochastic volatility achieves reduced impact of model
The remainder of this paper is organized as follows. In Section 2 we present our model assumptions for the dynamics of the spot energy prices. Section 3 provides a description of the data and the estimation methodology employed. Empirical results are presented in Section 4. In Section 5 we introduce our valuation framework for discrete arithmetic average options and apply in pricing and hedging. Section 6 concludes. Mathematical proofs are deferred to the appendix.

2 Model Specification and Properties

Let \((\Omega, \mathcal{F}, P)\) be a probability space equipped with filtration \(\mathbb{F} := (\mathcal{F}_t)_{t \geq 0}\). Define the commodity spot price process \(S\) as the sum of a predictable seasonal component and a stochastic component

\[ S_t = f_t + \exp(X_t), \quad t \geq 0. \]

The predictable component defined as

\[ f_t = \delta_0 t + \delta_1 \sin(t + 2\pi\tau_1/264) + \delta_2 \sin(t + 4\pi\tau_2/264), \]

with parameters \(\delta_0, \delta_1, \tau_1, \delta_2, \tau_2\), accounts for deterministic regularities in the spot price evolution. Geman (2005) suggests a mean-reverting model framework for commodity spot prices with stochastic volatility. We adopt the particular model specification and extend this to accommodate sudden jumps in the prices, so that the deseasonalized log-spot price \(X\) dynamics are given by

\[ dX_t = k(\varepsilon - X_t)dt + \sqrt{V_t}dB_t + dL_t, \]

where \(k\) is the speed at which random shocks dissipate and process \(X\) reverts toward the mean level \(\varepsilon\). The evolution of the spot price variance \(V\) is modelled by the square-root
diffusion as in Heston (1993), i.e.,

\[ dV_t = \alpha(\beta - V_t)dt + \gamma\sqrt{V_t}dW_t \]  

(3)

with positive parameters \( \alpha \) (speed of variance mean-reversion), \( \beta \) (long-run mean variance) and \( \gamma \) (volatility of variance). \( B \) and \( W \) are correlated standard Brownian motions (i.e., \( E(B_tW_t) = \rho t \)) allowing for possible inverse leverage effect, i.e., high prices associated with high volatility translating to \( \rho > 0 \). The spot price jump arrival is governed by the independent time-homogeneous compound Poisson process \( L \) with constant arrival rate of \( \lambda > 0 \) jumps (per unit time) of independent and identically normally distributed sizes \( J \) with \( E(J) =: \mu_J \) and \( \text{Var}(J) =: \sigma_J^2 \). Henceforth, we will be using the acronym MRJSV when referring to the model (2)–(3).

In this study we additionally consider three nested cases of the general model (2)–(3) depending on the assumptions on the spot price jumps and volatility:

MRSV: diffusion model (\( dL_t = 0 \) for all \( t \)) with Heston stochastic volatility.

MRJ: jump diffusion model with constant volatility \( \sqrt{V_t} = \sigma \) for all \( t \).

MR: Schwartz dynamics with \( dL_t = 0 \) and \( \sqrt{V_t} = \sigma \) for all \( t \).

In Sections 3 and 5 we derive futures and option prices based on our proposed model framework. For this purpose, we require risk neutral dynamics for the spot price. Following Benth (2011) we employ standard change of measure with respect to the Brownian motion driving the spot price dynamics

\[ d\tilde{B}_t := dB_t + \frac{h}{\sqrt{V_t}}dt, \]

where \( h \) is the (constant) market price of risk. Note that one may also introduce a change of measure in the volatility dynamics; we do not consider this here but, instead, as we explain
in Section 3, we calibrate the stochastic volatility model to market prices of traded futures contracts. A measure-change with respect to the jump process $L$ is also not considered.

MRJSV belongs to the class of affine-structure models (see Duffie et al., 2000 and Duffie et al., 2003), hence the characteristic function
\[ \phi_{V,X}(t,u_1,u_2) := E(\exp\{iu_1V_t + iu_2X_t\}) \]
has exponentially affine dependence on $V$ and $X$, i.e., there exist functions $\psi_0, \psi_1, \psi_2 : \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{C}$ so that
\[ \phi_{V,X}(t,u_1,u_2) = \exp\{\psi_J(t,u_2) + \psi_0(t,u_1,u_2) + \psi_1(t,u_1,u_2)V_0 + \psi_2(t,u_1,u_2)X_0\}, \quad (4) \]
where
\[ \psi_J(t,u_2) := \lambda t \left( e^{iu_2\mu_J - \frac{1}{2}\sigma_J^2 u_2^2} - 1 \right) \]
and $\psi_0, \psi_1, \psi_2$ satisfy the system of Riccati equations
\[ \frac{\partial \psi_2}{\partial t} = -k \psi_2, \quad (5) \]
\[ \frac{\partial \psi_1}{\partial t} = -\alpha \psi_1 + \frac{1}{2} \gamma_2 \psi_2^2 + \rho \gamma \psi_1 \psi_2 + \frac{1}{2} \psi_2^2, \quad (6) \]
\[ \frac{\partial \psi_0}{\partial t} = \alpha \beta \psi_1 + k(\varepsilon - h/k) \psi_2, \quad (7) \]
subject to the boundary conditions
\[ \psi_0(0,u_1,u_2) = 0, \quad \psi_1(0,u_1,u_2) = iu_1, \quad \psi_2(0,u_1,u_2) = iu_2. \]

By straightforward integration of (5) and (7) we get that
\[ \psi_2(t,u_1,u_2) = iu_2 e^{-kt} \quad (8) \]
and
\[ \psi_0(t,u_1,u_2) = \alpha \beta \int_0^t \psi_1(s,u_1,u_2)ds + iu_2(\varepsilon - h/k)(1 - e^{-kt}), \]
respectively. Solving \( (6) \) explicitly is not trivial; from Proposition \( \text{3} \) (see Appendix A.1), the general solution to \( (6) \) is given by \( (A.3) \)

\[
\psi_1(t, u_1, u_2) = -\frac{2iu_2k}{\gamma^2} e^{-kt} \zeta(iu_2 e^{-kt}) + \frac{M(t, u_2)}{C(u_1, u_2)} - \frac{1}{2} \int_0^t M(s, u_2) ds,
\]

where \( M \) is defined in \( (A.6) \) and

\[
C(u_1, u_2) := \frac{\exp \left( -\frac{iu_2\gamma}{k} + 2 \int_0^{iu_2} \zeta(y) dy \right)}{iu_1 + \frac{2iu_2k}{\gamma^2} \zeta(iu_2)}
\]
satisfies \( \psi_1(0, u_1, u_2) = iu_1 \). Function \( \zeta \), which satisfies Eq. \( (A.1) \), admits the representation

\[
\zeta(y) = \sum_{j=1}^{\infty} d_j y^j,
\]

where the coefficients \( \{d_j\}_{j=1}^{\infty} \) satisfy the recursion

\[
\left( j + 1 - \frac{\alpha - k}{k} \right) d_{j+1} = \sum_{i=1}^{j-1} d_i d_{j-i} 1_{j>1} - \frac{\beta \gamma}{k} d_j 1_{j>0} + \frac{\gamma^2}{4k^2} 1_{j=0}.
\]

(Alternatively, one may consider solving \( (6) \) numerically using, e.g., Matlab built-in solvers, however, at increased computational cost.)

In the case of the one-dimensional MRJ model with constant volatility \( \sigma \), the characteristic function \( \phi_X(t, u) := E \left( e^{iuX} \right) \) takes the simplified form

\[
\phi_X(t, u) = \exp \left\{ \psi_J(t, u) + iu(\varepsilon - h/k)(1 - e^{-kt}) - \frac{\sigma^2 u^2}{4k}(1 - e^{-2kt}) + iue^{-kt} X_0 \right\}
\]

(see Cont and Tankov, 2004), whereas in the cases of the MRSV and MR models the characteristic functions of \( (V, X) \) and \( X \), respectively, follow directly from \( (4) \) and \( (9) \) for \( \lambda = 0 \).
3 Data and Estimation Methodology

In this study we consider five energy commodities: two representatives of the European petroleum market and three from the US market. European spot prices are the dated Brent (CB), which serves as a benchmark assessment of the price of physical light crude oil in the North Sea, and the gasoil (GO) European Economic Community Cost in North West Europe; corresponding futures contracts are traded on the Intercontinental Exchange (ICE Futures Europe). US spot prices are the light sweet crude oil WTI (CL) at Cushing, Oklahoma, RBOB gasoline (HU), and No. 2 heating oil (HO) in New York harbor; corresponding futures are traded on NYMEX of the CME Group. For the purposes of our study we use a time series of constant-maturity futures; the futures curve is constructed by cubic spline interpolation of market prices of traded futures contracts. Volume and open interest data lead us to consider a block of 12 contracts for each commodity from 1 up to 12 months to maturity. The choice of constant-maturity contracts ensures that all prices are measured at the same point in time avoiding problems related to thin trading, expiration effects and discontinuities from rolling over futures contracts. We consider a 4-year period from March 12, 2009 to March 11, 2013, that is, a total of 1,043 daily observations after the necessary refinements for bank holidays. Data are collected from Datastream.

We use daily historical spot price data to estimate the seasonality function for each commodity. We then employ a two-stage estimation procedure based on Clewlow and Strickland (2000), Benth (2011) and Broadie et al. (2007). First, using the deseasonalized log-spot price time series, we obtain parameter estimates of the MRJSV and nested MRSV, MRJ, MR models of Section 2. In the second stage, given these parameters, we use the information embedded in end-of-day futures contract quotes to estimate the volatility parameters and the market price of risk.

In more details, we define a jump as an observation in the deseasonalized log-price changes series that is greater in absolute value than a market-specific threshold given by a multiple of the sample standard deviation of the deseasonalized series (see Clewlow and Strickland,
The prices on the identified ‘jump dates’ are substituted by the averages of the two adjacent prices, the standard deviation of the updated series is recalculated and the same procedure is repeated until no more jumps are identified. We then estimate the jump arrival rate $\lambda$ by the average number of identified jumps per year; the estimates of the mean $\mu_J$ and standard deviation $\sigma_J$ of the jump size distribution are given by the average and standard deviation of the jump returns, respectively; the log-spot price mean-reversion rate $k$ and the long-run mean log-spot price $\varepsilon$ are estimated using OLS regression. In order to calibrate the remaining parameters, i.e., the variance model parameters $V_0, \alpha, \beta, \gamma$, the correlation coefficient $\rho$ and the market price of risk $h$, we use end-of-day futures contract quotes. This is possible given explicit expressions for the futures price $F$ under the different model specifications.

**Proposition 1** The futures price $F(0, T)$ at time 0 for a contract expiring at time $T > 0$ is given by

$$F(0, T) = E(S_T) = f(T) + E(e^{X_T}),$$

where

$$E(e^{X_T}) = \begin{cases} 
\phi_{V, X}(t, -i), & \text{under MRJSV (and MRSV with } \lambda = 0) \\
\phi_X(t, -i), & \text{under MRJ (and MR with } \lambda = 0) 
\end{cases}.$$

**Proof.** Follows from (4) and (9). \qed

Let $F_{j,l}$ be the observed futures prices at time $t_j$ of a contract maturing at $T_l$. The theoretical futures price $F_\theta(t_j, T_l)$ is given in Proposition 1, where the subscript $\theta$ indicates dependence on the parameter vector $\theta := (V_0, \alpha, \beta, \gamma, \rho, h)$. We obtain $\theta$ by minimizing the Euclidean distance between the observed and theoretical futures prices

$$\theta^* := \arg\min_\theta \sum_{j=1}^n \sum_{l=1}^m \sqrt{|F_{j,l} - F_\theta(t_j, T_l)|}^2,$$

(10)

We choose the threshold (multiple $m$ of the sample standard deviation) that leads to the best calibrated model in terms of minimizing the Jarque and Bera (1980) statistic of the filtered series (ensures that daily log-price changes are closest to the normal distribution). To this end, we implement the jump-removal procedure for different multiples $m = 0.5 + k/100$, $k = 0, \ldots, 450$, and find that the optimal multiples $m^*$ are for each market 2.8 (CB), 2.75 (GO), 2.7 (CL), 2.7 (HU) and 2.8 (HO).
where \( m = 12 \) is the number of maturities and \( n = 1,043 \) the number of days in the sample.

4 Empirical Results

In this section we discuss the estimation results of the four model specifications MRJSV, MRSV, MRJ and MR for each of the CB, GO, CL, HU and HO markets. We begin the analysis with the calibration results. We then test the performance of the different spot models in commodity futures pricing. By means of a simulation study, we further examine if the specified models can accurately represent the true spot price dynamics.

4.1 Model Calibration

We fit first a sinusoidal function with a trend and two terms capturing annual and semi-annual seasonal cycles (see Eq. [1]). Results suggest that all markets exhibit significant seasonal patterns: adjusted \( \bar{R}^2 \) are 76% and 82% for the European CB and GO, and 59%, 79% and 83% for the US CL, HU and HO.

Next, we employ the estimation procedure described in Section 3. Panel A of Table 1 shows the model calibration outcome. Several remarks are in order. The mean-reversion parameter estimates imply that the expected time for the (log) spot prices to return half way toward the equilibrium level \( \varepsilon \) is 51 days (half-life \( k^{-1} \ln 2 \)), on average, for the European markets (CB and GO) and 35 days for the US markets (CL, HU and HO); inclusion of jumps in the spot price model results into increased half-lives of 55 and 44 days, respectively, as jump returns are less persistent. Jumps are infrequent events and tend to be negative. More specifically, we find that jumps in the European markets are less frequent, i.e., expected 3.2-3.4 jumps per year for GO and CB versus 4.6-5.0 for HO and CL and 7.6 for HU, and have smaller standard deviation, i.e., 6.4%-6.7% for CB and GO versus 7.7%-7.9% for CL and HO and 9.2% for HU. The relative importance of the jump component is also evident by its percentage contribution to the total variance of the fitted spot model, which lies in
12%-13% for CB and GO, 19%-21% for CL and HO, and 29% for HU.

Turning next to the volatility model parameters, in the case of the MRSV model the estimated speeds of variance mean-reversion translate to average half-lives ($\alpha^{-1} \ln 2$) of 5.2 (CB and GO) and 3.6 days (CL, HU and HO). In general, the long-run mean variance, volatility of variance and the correlation between log-prices and variance innovations are lower for the European CB and GO than for the US CL, HO and HU. It is worth pointing out that the correlation is found positive in consistency with the inverse leverage effect in energy prices, meaning that high prices are associated with high volatility, which can be attributed to the negative relationship between prices and inventory (e.g., see Pindyck, 2004). Incorporating price jumps (i.e., case of the MRJSV model) has a downward effect on the speed of variance mean-reversion, long-run mean variance, volatility of variance and correlation level, as the need for the variance process to create large sudden movements becomes less important.

In addition, we compute the root mean square error (RMSE) and relative RMSE (RRMSE) of the model-implied theoretical futures prices given in Proposition 1 for each model with respect to the observed futures prices: $\text{RMSE} := \sqrt{\frac{1}{nm} \sum_{j=1}^{n} \sum_{l=1}^{m} |F_{\theta^*}(t_j, T_l) - F_{j,l}|^2}$ and $\text{RRMSE} := \sqrt{\frac{1}{nm} \sum_{j=1}^{n} \sum_{l=1}^{m} |F_{\theta^*}(t_j, T_l)/F_{j,l} - 1|^2}$, where $F_{j,l}$ is the observed futures prices at time $t_j$ of a contract maturing at $T_l$, $F(t_j, T_l)$ is the theoretical futures price given in Proposition 1 for each model under the optimal parameter set $\theta^*$ (see Eq. 10), $m = 12$ (no. of maturities) and $n = 1,043$ (no. of days in the sample period March 12, 2009 to March 11, 2013). Results presented in Panel B of Table 1 suggest that MRJSV generates lowest pricing error: across markets, average RRMSE is 13% and maximum RRMSE is 16% (HU). (We reach similar conclusion if we consider the mean absolute percentage error $\text{MAPE} := \frac{1}{nm} \sum_{j=1}^{n} \sum_{l=1}^{m} |F_{\theta^*}(t_j, T_l)/F_{j,l} - 1|$; results are currently not presented for brevity.) To test whether any of the extended versions of the basic MR model with price jumps and/or stochastic volatility yields significant improvement in the futures pricing per-
formance, we apply the Hansen (2005) test\(^3\). We find that accurately modelling jumps in the spot price dynamics improves satisfactorily the fit of the resulting theoretical futures prices to the observed futures prices compared to the MR model; admitting additionally stochastic volatility improves the fit further. More specifically, compared to MR, MRJSV provides a 4% average, across markets, reduction in the RMSE. Unsurprisingly, highest reduction, that is, 6.3%, is reported in the HU market having the most frequent price jumps, highest jump size standard deviation, volatility of variance and correlation between variance and log-price processes. The improvement in the fit brought by MRJSV compared to MRSV is still high, that is, 2.1% average reduction in the RMSE, whereas compared to MRJ the RMSE reduction is up to 1%.

We conclude this analysis by referring to an additional error statistic, that is, the mean percentage error \(\text{MPE} := \frac{1}{nm} \sum_{j=1}^{n} \sum_{l=1}^{m} \left( F_{\theta^*}(t_j, T_l) / F_{j,l} - 1 \right) \), which we compute for the entire term structure of futures prices and use to test for systematic bias. The outcome of the test suggests rejection of this hypothesis at conventional significance levels. For brevity we do not detail these results here, but we can make these available upon request.

4.2 Simulation Study: True Data-Generating Process

As a next stage in our analysis, we examine whether the proposed models can accurately represent the true price dynamics of the commodities of interest. Similarly to Kaeck and Alexander (2013), we compare the model-implied distributions to the empirical one for each commodity in terms of a set of statistics. We employ the following test procedure: for each commodity we approximate the empirical distribution of the log-returns using stationary bootstrap and construct 90% bootstrap confidence intervals for the standard deviation, skewness, kurtosis, 1st and 99th percentiles and expected shortfalls at the 1% and 99% levels. Then, for each

\(^3\)We define loss function (LF) differentials between the MR model and each of the MRSV, MRJ and MRJSV models based on the RMSE and RRMSE error statistics, e.g., \(\text{LF}_{t_j} = \text{RMSE}_{\text{MRSV}, t_j} - \text{RMSE}_{\text{MRJSV}, t_j} \), where \(j = 1, \ldots, n = 1,043\). Then, using 5,000 bootstrap simulations, we test the null hypothesis that MR is not outperformed by the other models, i.e., \(H_0 : E(\text{LF}) \leq 0 \). For more details, we refer to White (2000) and Hansen (2005); a description of the stationary bootstrap algorithm can also be found in Politis and Romano (1994) and Sullivan et al. (1999, Appendix C).
model, we simulate 100,000 price paths and log-return sample statistics, and calculate the percentage number of simulated statistics of a given type falling within the corresponding bootstrap confidence interval; a value closer to 1 implies smaller discrepancy between the observed and assumed price dynamics. The model with the best relative performance in terms of given statistic type is indicated by an asterisk in Table 2. If we take a close look at the table, we see that, although the MR model appears to be fitting well the standard deviation and the 99th percentile (65% and 89%, respectively, of the time on average across markets), it naturally does not capture the skewness and kurtosis (33% and 10%, respectively) of the empirical distribution. As a result, the left tail is missed more often in the case of MR (and MRSV but to a lesser extent). On the other hand, including jumps (i.e., case of MRJ and MRJSV models) allows for a more balanced and thus accurate fitting of the tails, adequately accounting for both the skewness and excess kurtosis commonly found in commodity log-returns. In particular, comparing model-implied standard deviation, skewness and kurtosis across the five markets for each model, we observe that MRJSV produces more realistic statistics in 10 cases out of 15 (3 statistics × 5 markets). The 1st and 99th percentiles as well as the expected shortfalls at the 1% and 99% levels of the empirical and model-implied log-return distributions are also reported suggesting overall superiority of the MRJSV model in 10 out of 20 cases.

In addition, we employ the two-sample Kolmogorov–Smirnov test with the null hypothesis stating that the observed and simulated log-return distributions are equal. To this end, for each of the simulated price paths of the competing models, we test the null hypothesis and store the p-value. Table 2 reports the percentage number of times the null hypothesis cannot be rejected at the 10% significance level. In the case of the MR model the null hypothesis cannot be rejected in 62.1% (CB), 56.5% (GO), 55.1% (CL), 35.7% (HO) and 59.7% (HU) of the simulation runs. Allowing for price jumps or stochastic volatility (i.e., MRJ or MRSV model) raises the evidence in favour of the null hypothesis, while the combined effect (i.e., case of MRJSV model) is found strongest with non-rejection in 88.4% (CB), 92.8% (GO),
82.1% (CL), 84.4% (HO) and 80.1% (HU) of the time.

Overall, we find that the MRJSV model is sufficiently rich and capable of providing a more realistic description of the true DGP, accounting for the stylized patterns of the spot prices in the petroleum markets under consideration.

5 Application on Discrete Arithmetic Asian Options

In what follows we investigate the impact of the proposed stochastic models in Section 2 in terms of the pricing and delta hedging of options on the discrete arithmetic average spot price, which are particularly popular in commodity markets. The need for realistic spot price models capturing the unique characteristics of energy commodities raises substantially the complexity of the option pricing problem. We solve this by means of an efficient Monte Carlo simulation approach, which we develop based on the model properties presented in Section 2. We then execute pricing and delta hedging simulation exercises based on the calibrated model parameters in Section 4.

5.1 Option Pricing

Suppose that the spot price \( S \) is monitored over the period \([0, T]\), \( T > 0 \), at the following equidistant dates: \( 0, \delta, \ldots, j\delta, \ldots, n\delta = T \).

The terminal payoff of the arithmetic Asian option depends on the average of the past \( n + 1 \) spot prices

\[
\frac{1}{n + 1} \sum_{j=0}^{n} S_{j\delta} = A_{n+1} + \frac{1}{n + 1} \sum_{j=0}^{n} f_{j\delta},
\]

where \( f \) is given by (1) and

\[
A_{n+1} := \frac{1}{n + 1} \sum_{j=0}^{n} (S_{j\delta} - f_{j\delta}) = \frac{1}{n + 1} \sum_{j=0}^{n} e^{X_{j\delta}}
\]

is the average deseasonalized spot price. The price of the arithmetic Asian call option with
fixed strike price $K$ reads

$$E \left[ e^{-rT} \left( \frac{1}{n+1} \sum_{j=0}^{n} S_{j\delta} - K \right)^+ \right] = E \left[ e^{-rT} \left( A_{n+1} - \tilde{K} \right)^+ \right],$$  \hspace{1cm} (11)

where $x^+ := \max(x, 0)$,

$$\tilde{K} := K - \sum_{j=0}^{n} f_{j\delta}/(n+1)$$

is the deseasonalized strike price, and $r$ the continuously compounded risk free rate of interest. Note that the price of the put-type option can be obtained via standard put-call parity. By analogy, the price of the Asian call option on the geometric average spot price is given by

$$E \left[ e^{-rT} \left( e^{Y_{n+1}} - \tilde{K} \right)^+ \right],$$  \hspace{1cm} (12)

where

$$Y_{n+1} := \frac{1}{n+1} \sum_{j=0}^{n} \ln(S_{j\delta} - f_{j\delta}) = \frac{1}{n+1} \sum_{j=0}^{n} X_{j\delta}.$$  \hspace{1cm} (13)

For the purposes of our analysis, we estimate the price of the arithmetic option, i.e., evaluate (11), by employing an accurate control variate Monte Carlo (CVMC) scheme with 100,000 simulations and the geometric Asian option price (12) used as control variate. Given the exact closed-form expression for the price of the geometric option under Black–Scholes–Merton model assumptions, Kemna and Vorst (1990) show that the geometric option serves as an efficient control variate in the simulation of the arithmetic option. Fusai and Meucci (2008) extend to Lévy log-spot models by adopting a Fourier transform approach, whereas in this paper we generalize to non-Lévy members of the affine class, including the MRJSV model and the nested models of Section 2. To this end, we derive first the characteristic function of the log-geometric average, $\phi_{Y_{n+1}}(u) := E\{\exp\{iuY_{n+1}\}\}$, under the given model assumptions.

**Proposition 2** The characteristic function of the log-geometric average $Y_{n+1}$ defined in (13) is given:
(a) under the MRJSV model dynamics by

$$\phi_{Y_{n+1}}(u) = \exp \left\{ \sum_{j=1}^{n} (\psi_J(\delta, \vartheta_j) + \psi_0(\delta, \eta_j, \vartheta_j)) + \psi_1(\delta, \eta_1, \vartheta_1)X_0 + \psi_2(\delta, \eta_1, \vartheta_1)X_0 \right\},$$

(14)

where $\eta_n := 0$, $\vartheta_n := u/(n+1)$, $\eta_j := -i\psi_1(\delta, \eta_{j+1}, \vartheta_{j+1})$ and $\vartheta_j := -i\psi_2(\delta, \eta_{j+1}, \vartheta_{j+1}) + u/(n+1)$ for $0 < j \leq n-1$.

(b) under the MRJ model dynamics by

$$\phi_{Y_{n+1}}(u) = \prod_{j=0}^{n} \exp \left\{ \frac{iu}{n+1}X_0e^{-kj\delta} \right\} \times \prod_{j=1}^{n} \exp \left\{ \psi_J(\delta, \eta_j) + i\eta_j(\varepsilon - h/k)(1 - e^{-k\delta}) - \frac{\sigma^2\eta_j^2}{4k}(1 - e^{-2k\delta}) \right\},$$

where $\eta_j := u \sum_{m=0}^{n-j} e^{-mk\delta} / (n+1)$ for $0 < j \leq n$.

Relevant results under the MRSV and MR models follow directly by setting $\lambda = 0$.

**Proof.** See Appendix A.2. ■

Given the characteristic function of the log-geometric average, the exact price of the geometric Asian option \[12\] can be computed by means of the Fourier-inversion formula with respect to the log-strike price $\tilde{\kappa} := \log \tilde{K}$

$$E \left[ e^{-rT} (e^{Y_{n+1}} - \tilde{K})^+ \right] = \frac{e^{-\xi\tilde{\kappa} - rT}}{2\pi} \int_{\mathbb{R}} e^{-iu\tilde{\kappa}} \frac{\phi_{Y_{n+1}}(u - \xi i - i)}{(iu + \xi)(iu + \xi + 1)} du,$$

(15)

where constant $\xi > 0$ ensures integrability in $\tilde{\kappa}$ (see Carr and Madan, 1999). Formula (15) can be evaluated using the (fractional) fast Fourier transform algorithm (e.g., fft or czt in Matlab, see Černý and Kyriakou, 2011 for more details) which outputs the geometric Asian option prices on a fine, equally spaced grid of log-strike prices.

Based on the estimated model parameters (see Table 1) and using our results for the geometric Asian option, we implement standard CVMC setup (see Table 3 and Glasserman, 2004 for more details) to estimate the prices of at-the-money (ATM), in-the-money (ITM)
and out-of-the-money (OTM) 1-month to maturity arithmetic Asian call options with daily monitoring (i.e., \( n = 22 \)). In particular, the strike price is set equal to 100\%, 95\% and 105\% of the spot price. The current spot price and variance are set equal to their long-run mean levels and the risk free interest rate is given by the average 3-month US T-bill rate throughout the sample period. Computed option prices are presented in Table 4 in monetary terms (CB - $/bbl, GO - $/mt, CL - $/bbl, HU - c$/gal, HO - c$/gal) as well as relative to the spot price and the price of the corresponding ATM option.

Results suggest that ignoring jumps and stochastic volatility leads to lower option prices. More specifically, we find that, across all markets, MRSV and MRJSV prices are systematically higher than the corresponding MRJ prices, which, further, are higher than the MR ones. In fact, the discrepancies between MRSV, MRJSV and MRJ prices versus MR prices are higher for in-the-money options, whereas these reduce the more out-of-the-money the option.

Our observation based on option prices under non-Lévy (i.e., dependent) log-returns adds to earlier contributions by Fusai and Meucci (2008) and Černý and Kyriakou (2011), who observe similar pattern across different level of moneyness based on option prices relying on Lévy (i.e., independent) assumptions for stock log-returns. As noted in Černý and Kyriakou (2011), such pattern is attributed to the combined kurtosis-skewness effect in a Lévy and non-Lévy, as opposed to normal, log-return distribution.

### 5.2 Hedge Performance Comparisons

The MRJ, MRSV and MRJSV models we have considered in this paper generally correspond to incomplete markets due to the presence of stochastic jumps and/or volatility, which introduces a hedging error in a delta hedging strategy, even in the theoretical limit of continuous rebalancing. In the case of one-factor diffusion dynamics, like MR, existence of error can be attributed to discreteness in rebalancing frequency and transactions costs. In what follows we adhere to the common market practice of delta hedging and apply to the case of Asian options. We define the hedging error as the difference between the value of the hedge port-
folio and the value of a long 1-month to maturity ATM arithmetic Asian option with daily monitoring for a 1-week hedge period.

For each commodity we simulate the distribution of the hedging error when the underlying data-generating process is given by either of the MR, MRJ, MRSV and MRJSV models and (1) the hedge portfolios are formed accordingly, or (2) the hedge portfolios are misspecified, i.e., are formed based on alternative models. In fact, the second case is the most realistic one as the hedger never really knows the true DGP and, hence, develops the hedging approach based on some assumptions. The particular task allows us to investigate the sensitivity of the hedging performance to model misspecification, in other words, the impact of incorrectly specifying the underlying data-generating process and forming the hedge portfolios. In simulating the hedging error distributions, the option prices are estimated using the CVMC approach in Section 5.1 whereas the option deltas using a modified version based on straightforward application of the pathwise technique (see Glasserman, 2004) with the geometric Asian option delta used as control variate.

Table 5 reports for each market the mean and standard deviation of the hedging error for given driving process and hedge portfolios formed accordingly (see Panels A.1, B.1, C.1, D.1). The mean hedging error is found negative for all markets. Models with stochastic volatility generate hedging errors with largest standard deviation. This is reflected in Figure 1 which depicts the model-implied distributions of the log-returns under a MR or MRJSV DGP and the corresponding hedging error distributions; the daily movements under the MR model are relatively small, whereas the MRJSV model generates log-return distributions with fat-tails impacting the hedging performance. Overall, hedges are sensitive to a change from the MR to MRJSV driving process.

In addition, Table 5 reports the percentage changes in the standard deviation of the hedging error under different misspecified hedge portfolios (see Panels A.2, B.2, C.2, D.2), which allow for an inspection of the effect of model uncertainty on the hedging error dis-

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4For brevity we present only the two extreme model cases; additional plots can be made available upon request.
tribution. Negative entries indicate that the misspecified model performs better than the supposed DGP as it results in reduced standard deviation of the hedging error. In general, as mentioned, for example, in Carr and Wu (2013), under practical situations when the true underlying price dynamics are unknown to the hedger, it may be the case that a misspecified hedge works as well or better than some other. For example, supposing that MR is the driving process, implementing instead a MRJSV hedge portfolio leads to reduced standard deviation of the hedging error by 0.04% (CL) to 8.12% (HU). In fact, we observe that, for any chosen driving process, the MRJSV hedge portfolio always achieves reduced standard deviation (see negative entries). Across different environments, largest reduction is observed for the HU market. Finally, from Panel D of Table 5 where the MRJSV model represents the true DGP, which is also corroborated by our analysis in Section 4, we find that model misspecification always leads to deteriorated hedging performance (only positive entries); we note, in this case, that the MRSV hedge generates the least performance deterioration (excl. CB), which is sensible given that the MRSV versus MRJSV option price differences are the smallest ones (see ATM, Table 4).

6 Summary and Conclusion

We have considered five major European and US petroleum commodity markets to study the ability of four different model specifications to realistically portray the dynamics of the data-generating process using historical spot and futures price data over the period March 2009 to March 2013. Our selection of models includes a seasonal mean-reverting spot price model with jumps and Heston-type stochastic volatility (MRJSV) and nested specifications with/out jumps or stochastic volatility (MR, MRJ, MRSV). In the first stage of our analysis, we assess relative performance by comparing simulated model-implied and empirically estimated statistics as well as testing for equality of the observed and simulated log-return distributions; our exercise indicates superior performance of the MRJSV model
in producing dynamics that are similar to the true price dynamics. Furthermore, using the theoretical futures prices derived for all competing models, we investigate the fit to the observed futures curves in the different markets; our error statistics point toward inclusion of price jumps, while combining with stochastic price volatility improves the fit further.

In addition, given the widespread use of Asian options in commodity markets, we provide an efficient valuation framework for the prevalent case of the arithmetic average with discrete monitoring when the underlying spot is driven by a general exponential affine model. Our findings suggest that failing to account for price jumps and stochastic volatility leads to relatively lower option premia, especially for in-the-money options. Furthermore, we execute a delta hedging exercise where we investigate the sensitivity of the hedging performance to model misspecification. An interesting conclusion from this exercise is that a misspecified hedge may perform better than a hedge formed under the supposed DGP. In fact, we find that, for different possible representations of the true DGP, the MRJSV hedge systematically achieves reduced standard deviation of hedging error in all markets, adding to our earlier conclusions in support of the MRJSV paradigm.

The implications of this research are critical for market participants, as specifying correctly the dynamic behaviour of the underlying spot price process is important for understanding and managing the risks associated with derivative securities, investment evaluation, asset allocation and planning.

Appendix A: Proofs

A.1 Proof of general solution to Riccati equation (6)

Proposition 3 Suppose \( \zeta(y), y \in \mathbb{C}\{0\}, \) satisfies the equation

\[
\zeta'(y) = \zeta(y)^2 + \left( \frac{\alpha - k}{ky} - \frac{\rho \gamma}{k} \right) \zeta(y) + \frac{\gamma^2}{4k^2}.
\]  
(A.1)
Define
\[ \chi(t, u_2) = -\frac{2iu_2k}{\gamma^2} e^{-kt} \zeta(iu_2 e^{-kt}). \]  
(A.2)

Then,

(a) \( \chi \) is a solution to (6).

(b) the general solution to (6) takes the form

\[ \psi_1 = \chi + \frac{1}{w}, \]  
(A.3)

where \( w \) satisfies the differential equation

\[ \frac{\partial w}{\partial t} + (-\alpha + iu_2 \rho \gamma e^{-kt} + \gamma^2 \chi) w = -\frac{1}{2} \gamma^2. \]  
(A.4)

Proof.

(a) From (A.2), we obtain

\[
\frac{\partial \chi}{\partial t} = -k \chi - \frac{2u_2^2 k^2}{\gamma^2} e^{-2kt} \zeta'(iu_2 e^{-kt})
= -k \chi - \frac{2u_2^2 k^2}{\gamma^2} e^{-2kt} \left[ \zeta(iu_2 e^{-kt})^2 + \left( \frac{\alpha - k}{iu_2 k e^{-kt}} - \frac{\rho \gamma}{k} \right) \zeta(iu_2 e^{-kt}) + \frac{\gamma^2}{4k^2} \right]
= -\alpha \chi + \frac{1}{2} \gamma^2 \chi^2 + \rho \gamma \chi \psi_2 + \frac{1}{2} \psi_2^2,
\]  
(A.5)

where the second equality follows from (A.1) and \( \psi_2 \) in the last equality is given by (8). It is implied from (A.5) that \( \chi \) is a solution to (6).

(b) Suppose that \( \chi + 1/w \) is an arbitrary solution to (6). Then we have

\[
\frac{\partial \chi}{\partial t} - \frac{1}{w^2} \frac{\partial w}{\partial t} = -\alpha \left( \chi + \frac{1}{w} \right) + \frac{1}{2} \gamma^2 \left( \chi^2 + \frac{2\chi}{w} + \frac{1}{w^2} \right) + \rho \gamma \left( \chi + \frac{1}{w} \right) \psi_2 + \frac{1}{2} \psi_2^2.
\]
and, further, from (A.5) we get

\[- \frac{1}{w^2} \frac{\partial w}{\partial t} = -\frac{\alpha}{w} + \frac{1}{2} \gamma^2 \left( \frac{2\chi}{w} + \frac{1}{w^2} \right) + \frac{\rho \gamma \psi_2}{w},\]

from which (A.4) follows.

For the differential equation (A.4) with \(\chi\) given in (A.2), from standard calculus theory the integrating factor is

\[
M(t) := \exp \left( -\alpha t - \frac{iu_2 \rho \gamma}{k} e^{-kt} - 2iu_2 k \int e^{-kt} \zeta(iu_2 e^{-kt}) dt \right)
= \exp \left( -\alpha t - \frac{iu_2 \rho \gamma}{k} e^{-kt} + 2 \int_{0}^{e^{-kt}} \zeta(y) dy \right),
\]

(A.6)

from which we get the general solution to (A.4)

\[
w(t) = C - \frac{1}{2} \gamma^2 \int_{0}^{t} M(s) ds \]

where \(C\) is the constant of integration.

A.2 Proof of Proposition 2

Proof.

(a) Under the MRJSV model dynamics, we have from (1) that

\[
E \left( \exp \{ i\eta_n V_{n\delta} + i\vartheta_n X_{n\delta} \} | \mathcal{F}_{n-1}\delta \right)
= \exp \left\{ \psi_j(\delta, \vartheta_n) + \psi_0(\delta, \eta_n, \vartheta_n) + \psi_1(\delta, \eta_n, \vartheta_n) V_{n-1}\delta + \psi_2(\delta, \eta_n, \vartheta_n) X_{n-1}\delta \right\}, \quad (A.7)
\]
and for \( j = n - 1, n - 2, \ldots, 1 \)

\[
E \left( \exp \left\{ \psi_1(\delta, \eta_{j+1}, \vartheta_{j+1})V_{j\delta} + \left( \psi_2(\delta, \eta_{j+1}, \vartheta_{j+1}) + \frac{iu}{n + 1} \right) X_{j\delta} \right\} \bigg| \mathcal{F}_{(j-1)\delta} \right)
= \exp \left\{ \psi_j(\delta, \vartheta_j) + \psi_0(\delta, \eta_j, \vartheta_j) + \psi_1(\delta, \eta_j, \vartheta_j) V_{(j-1)\delta} + \psi_2(\delta, \eta_j, \vartheta_j) X_{(j-1)\delta} \right\}. \tag{A.8}
\]

Evaluating

\[
E(e^{iuY_{n+1}}) = E \left( \exp \left\{ \frac{iu}{n + 1} \sum_{j=0}^{n} X_{j\delta} \right\} \right)
\]

using iterated expectations and applying equalities (A.7)–(A.8), we obtain (14).

(b) Define \( Z_j = X_{j\delta} - X_{(j-1)\delta} e^{-k\delta} \), where \( j = 1, \ldots, n \). By recursive substitution in \( X_{j\delta} = X_{(j-1)\delta} e^{-k\delta} + Z_j \) and summation across all \( j \), we get

\[
\sum_{j=0}^{n} X_{j\delta} = X_0 + \sum_{j=1}^{n} \left\{ X_0 e^{-kj\delta} + \sum_{m=0}^{n-j} Z_j e^{-mk\delta} \right\}.
\]

Then,

\[
E(e^{iuY_{n+1}}) = E \left( \exp \left\{ \frac{iu}{n + 1} \sum_{j=0}^{n} X_{j\delta} \right\} \right)
= \exp \left\{ \frac{iu}{n + 1} \sum_{j=0}^{n} X_0 e^{-kj\delta} \right\} E \left( \exp \left\{ \frac{iu}{n + 1} \sum_{j=1}^{n} Z_j \sum_{m=0}^{n-j} e^{-mk\delta} \right\} \right)
= \prod_{j=0}^{n} \exp \left\{ \frac{iu}{n + 1} X_0 e^{-kj\delta} \right\} \prod_{j=1}^{n} E \left( \exp \{ i\eta_j Z_j \} \right),
\]

where the last equality follows by stochastic independence of the variables \( \{Z_j\}_{j=1}^{n} \) and the characteristic function of \( Z_j \) from Eq. (9).

\[
\]

References


This table presents the model calibration results. Panel A reports the estimated annualized parameters of the MRSV and MRJSV models for each of the Brent Crude Oil (CB), Gasoil (GO), WTI Crude Oil (CL), Gasoline (HU) and Heating Oil (HO) markets. Panel B reports the error statistics computed for the entire term structure of futures prices: 

\[
\text{RRMSE} := \sqrt{\frac{1}{mn} \sum_{j=1}^{n} \sum_{l=1}^{m} |F_\theta^*(t_j, T_l) - F_{j,l}|^2}
\]

and 

\[
\text{RMSE} := \sqrt{\frac{1}{mn} \sum_{j=1}^{n} \sum_{l=1}^{m} |F_\theta^*(t_j, T_l) - F_{j,l}|^2},
\]

where \(F_{j,l}\) is the observed futures prices at time \(t_j\) of a contract maturing at \(T_l\), \(F_\theta^*(t_j, T_l)\) is the theoretical futures price given in Proposition 1 for each model under optimal parameter set \(\theta^*\) (see Eq. 10), \(m = 12\) (no. of maturities) and \(n = 1,043\) (no. of days in sample period March 12, 2009 to March 11, 2013). In addition, we test the null hypothesis that none of MRJ, MRSV and MRJSV models leads to reduction in futures pricing errors (RRMSE and RMSE) relative to the MR model, by employing the Hansen (2005) test and the stationary bootstrap of Politis and Romano (1994) using 5,000 bootstrap simulations. Asterisks (*) indicate significance at the 10% level.

### Table 1: Model calibration

<table>
<thead>
<tr>
<th></th>
<th>CB</th>
<th>GO</th>
<th>CL</th>
<th>HU</th>
<th>HO</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Panel A: parameter estimates</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\varepsilon)</td>
<td>4.252</td>
<td>4.253</td>
<td>6.318</td>
<td>6.319</td>
<td>4.279</td>
</tr>
<tr>
<td>(\lambda)</td>
<td>-3.400</td>
<td>-3.200</td>
<td>-5.000</td>
<td>-7.600</td>
<td>-4.600</td>
</tr>
<tr>
<td>(\mu_J)</td>
<td>-0.026</td>
<td></td>
<td>-0.013</td>
<td></td>
<td>-0.002</td>
</tr>
<tr>
<td>(\sigma_J)</td>
<td>0.064</td>
<td>0.067</td>
<td>-0.077</td>
<td></td>
<td>0.092</td>
</tr>
<tr>
<td>(h)</td>
<td>0.599</td>
<td>0.389</td>
<td>0.643</td>
<td>0.533</td>
<td>0.223</td>
</tr>
<tr>
<td>(\alpha)</td>
<td>36.68</td>
<td>31.52</td>
<td>30.71</td>
<td>16.76</td>
<td>37.33</td>
</tr>
<tr>
<td>(\beta)</td>
<td>0.257</td>
<td>0.231</td>
<td>0.349</td>
<td>0.245</td>
<td>0.315</td>
</tr>
<tr>
<td>(\gamma)</td>
<td>1.259</td>
<td>0.989</td>
<td>1.860</td>
<td>1.084</td>
<td>1.531</td>
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<td>(\rho)</td>
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<td>0.181</td>
<td>0.254</td>
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<td>(V_0)</td>
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<td>0.186</td>
<td>0.262</td>
<td>0.196</td>
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<td><strong>Panel B: futures contracts pricing errors</strong></td>
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<td>RRMS</td>
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<td>7.782</td>
<td>0.146</td>
<td>63.18</td>
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<td></td>
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<td>0.131*</td>
<td>7.508*</td>
<td>0.142*</td>
<td>61.43*</td>
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<tr>
<td></td>
<td>MRSV</td>
<td>0.135</td>
<td>7.767</td>
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<td></td>
<td>MRJSV</td>
<td>0.129*</td>
<td>7.499*</td>
<td>0.140*</td>
<td>61.33*</td>
</tr>
</tbody>
</table>
Table 2: True data-generating process testing

We test whether the proposed models (MR, MRJ, MRSV, MRJSV) can accurately represent the true price dynamics of each commodity (CB, GO, CL, HU, HO). The table reports the relative performance across models in terms of percentage number of simulated log-return statistics of a given type lying within the corresponding 90% bootstrap confidence interval of the empirical statistic. Table entries correspond to values in the range 0 to 1: e.g., a value of 0.750 indicates that in 75% of 100,000 simulation runs, the simulated statistic has been within the bootstrap confidence interval. Abbreviations: standard deviation (std), skewness (skew), kurtosis (kurt), 1st and 99th percentiles (perc1 and perc99) and expected shortfalls at the 1% and 99% levels (ES1 and ES99). In addition, for each simulation, we employ the two-sample Kolmogorov–Smirnov (K–S) test for equality of the empirical and model-implied log-return distributions and report the percentage number of times the null hypothesis cannot be rejected at the 10% significance level. Asterisks (*) highlight best relative performance across models.

<table>
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<th></th>
<th>std</th>
<th>skew</th>
<th>kurt</th>
<th>perc1</th>
<th>perc99</th>
<th>ES1</th>
<th>ES99</th>
<th>K–S test</th>
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<tr>
<td>MR</td>
<td>0.731</td>
<td>0.031</td>
<td>0.295</td>
<td>0.280</td>
<td>0.890</td>
<td>0.144</td>
<td>0.803*</td>
<td>0.621</td>
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<td>MRJ</td>
<td>0.866</td>
<td>0.556*</td>
<td>0.513</td>
<td>0.708</td>
<td>0.951*</td>
<td>0.779</td>
<td>0.564</td>
<td>0.833</td>
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<tr>
<td>MRSV</td>
<td>0.789</td>
<td>0.223</td>
<td>0.963*</td>
<td>0.388</td>
<td>0.766</td>
<td>0.242</td>
<td>0.747</td>
<td>0.646</td>
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<tr>
<td>MRJSV</td>
<td>0.871*</td>
<td>0.515</td>
<td>0.787</td>
<td>0.744*</td>
<td>0.860</td>
<td>0.857*</td>
<td>0.509</td>
<td>0.884*</td>
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<td><strong>Gasoil (GO)</strong></td>
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<td></td>
<td></td>
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<td></td>
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</tr>
<tr>
<td>MR</td>
<td>0.575</td>
<td>0.122</td>
<td>0.189</td>
<td>0.263</td>
<td>0.817</td>
<td>0.109</td>
<td>0.881*</td>
<td>0.565</td>
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<tr>
<td>MRJ</td>
<td>0.896*</td>
<td>0.556</td>
<td>0.530</td>
<td>0.608</td>
<td>0.937*</td>
<td>0.777</td>
<td>0.772</td>
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<tr>
<td>MRSV</td>
<td>0.788</td>
<td>0.577</td>
<td>0.720</td>
<td>0.282</td>
<td>0.862</td>
<td>0.266</td>
<td>0.857</td>
<td>0.724</td>
</tr>
<tr>
<td>MRJSV</td>
<td>0.767</td>
<td>0.606*</td>
<td>0.772*</td>
<td>0.696*</td>
<td>0.821</td>
<td>0.800*</td>
<td>0.786</td>
<td>0.928*</td>
</tr>
<tr>
<td><strong>WTI Crude Oil (CL)</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>MR</td>
<td>0.679</td>
<td>0.479</td>
<td>0.000</td>
<td>0.243</td>
<td>0.834</td>
<td>0.003</td>
<td>0.341</td>
<td>0.551</td>
</tr>
<tr>
<td>MRJ</td>
<td>0.855</td>
<td>0.474</td>
<td>0.222</td>
<td>0.621</td>
<td>0.868*</td>
<td>0.728*</td>
<td>0.763</td>
<td>0.770</td>
</tr>
<tr>
<td>MRSV</td>
<td>0.836</td>
<td>0.428</td>
<td>0.283</td>
<td>0.442</td>
<td>0.836</td>
<td>0.455</td>
<td>0.970*</td>
<td>0.619</td>
</tr>
<tr>
<td>MRJSV</td>
<td>0.876*</td>
<td>0.636*</td>
<td>0.446*</td>
<td>0.737*</td>
<td>0.829</td>
<td>0.693</td>
<td>0.901</td>
<td>0.821*</td>
</tr>
<tr>
<td><strong>Gasoline (HU)</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>MR</td>
<td>0.733</td>
<td>0.515</td>
<td>0.000</td>
<td>0.062</td>
<td>0.970*</td>
<td>0.009</td>
<td>0.204</td>
<td>0.597</td>
</tr>
<tr>
<td>MRJ</td>
<td>0.824</td>
<td>0.559</td>
<td>0.352</td>
<td>0.597</td>
<td>0.932</td>
<td>0.804</td>
<td>0.743</td>
<td>0.760</td>
</tr>
<tr>
<td>MRSV</td>
<td>0.788</td>
<td>0.595</td>
<td>0.814*</td>
<td>0.242</td>
<td>0.752</td>
<td>0.426</td>
<td>0.656</td>
<td>0.685</td>
</tr>
<tr>
<td>MRJSV</td>
<td>0.855*</td>
<td>0.755*</td>
<td>0.577</td>
<td>0.719*</td>
<td>0.908</td>
<td>0.819*</td>
<td>0.918*</td>
<td>0.801*</td>
</tr>
<tr>
<td><strong>Heating Oil (HO)</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>MR</td>
<td>0.514</td>
<td>0.487</td>
<td>0.000</td>
<td>0.213</td>
<td>0.937</td>
<td>0.004</td>
<td>0.408</td>
<td>0.357</td>
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<tr>
<td>MRJ</td>
<td>0.872</td>
<td>0.578</td>
<td>0.481*</td>
<td>0.568</td>
<td>0.974</td>
<td>0.642</td>
<td>0.846</td>
<td>0.804</td>
</tr>
<tr>
<td>MRSV</td>
<td>0.781</td>
<td>0.442</td>
<td>0.364</td>
<td>0.888*</td>
<td>0.956</td>
<td>0.548</td>
<td>0.948*</td>
<td>0.817</td>
</tr>
<tr>
<td>MRJSV</td>
<td>0.892*</td>
<td>0.701*</td>
<td>0.465</td>
<td>0.872</td>
<td>0.996*</td>
<td>0.672*</td>
<td>0.941</td>
<td>0.844*</td>
</tr>
</tbody>
</table>
Table 3: Control variate Monte Carlo (CVMC) simulation scheme

Summarized CVMC simulation scheme. **Inputs:** $M$: number of simulations; $n$: number of monitoring dates; $T$: option time to maturity; $\delta$: time spacing; $\tilde{K}$: deseasonalized strike price; $r$: continuously compounded risk free interest rate; spot price model params.: $X_0$, $\varepsilon$, $k$, $\sigma$, $\mu$, $\sigma$, $h$; stoch. variance model params.: $\beta$, $\gamma$, $\rho$; $E(G) := E(e^{-rT}(\exp(\sum_{j=0}^{n} X_j/(n+1))−\tilde{K})^+$): exact price of geometric Asian option (pre-computed using formula [15]); $b^*$: optimal CV coefficient (pre-estimated using a pilot run, see Glasserman [2004]). **Control Variate Monte Carlo:** simulation of MR model (step 1.b): exact method, e.g., see Glasserman [2004]; simulation of square-root diffusion (step 1.c): $X_{ij}^2(\epsilon V_{j-1})$ noncentral chi-square random variable with $df := 4\alpha \gamma^{-2}$ degrees of freedom and noncentrality parameter $cV_{j-1}$, $c := 4\alpha e^{-\alpha \delta} \gamma^{-2} (1 - e^{-\alpha \delta})^{-1}$, is approximated using Andersen (2008) low-bias quadratic-exponential method; simulation of MRSV model (step 1.d): modified Andersen (2008) method with central discretization employed in both integrated log-spot price and variance; simulation of compound Poisson process (step 2): improved algorithm in Cont and Tankov (2004, Section 6.1, Algorithm 6.2).

<table>
<thead>
<tr>
<th>Inputs: $M$; $n$; $T$; $\delta \leftarrow T/n$; $K$; $r$; model params.; $E(G)$; $b^*$</th>
<th>CVMC simulation:</th>
</tr>
</thead>
<tbody>
<tr>
<td>1a. Generate $n$ indep. normal variates $N_{0j} \sim N(0,1)$ for $j = 1, \ldots, n$</td>
<td>1. Generate deseas. log-spot price (&amp; stochastic variance) sample path:</td>
</tr>
<tr>
<td>1b. Set $X_j \leftarrow X_{j-1} e^{kh} + (\varepsilon - h/k)(1 - e^{-kh}) + \sigma \sqrt{(1 - e^{-2kh})/(2k)} N_{01,j}$ (MR)</td>
<td>2. Generate independent compound Poisson process:</td>
</tr>
<tr>
<td>1c. Set $V_j \leftarrow c^{-1} e^{-\alpha \delta} X_{ij}^2(cV_{j-1})$ for $j = 1, \ldots, n$</td>
<td>2a. Generate $n$ indep. Poisson variates $N_{ij} \sim Pois(\lambda \delta)$ for $j = 1, \ldots, n$</td>
</tr>
<tr>
<td>1d. Set $X_j \leftarrow [X_{j-1}(1-k\delta/2) + (k\varepsilon - h - \alpha \beta \rho / \gamma) \delta + V_{j-1}(\alpha \delta / 2 - 1) \rho / \gamma$</td>
<td>2b. Generate $N_{ij}$ indep. normal variates $J_{j,i} \sim N(\mu, \sigma_j^2)$ for $i = 1, \ldots, N_{ij}$</td>
</tr>
<tr>
<td>+ $V_j(\alpha \delta / 2 + 1) \rho / \gamma + \sqrt{(1 - \rho^2)(V_{j-1} + V_j) \delta / 2 N_{01,j}}] (1 + k \delta / 2)^{-1}$ (MRSV)</td>
<td>2c. Generate $N_{ij}$ indep. uniform variates $U_{j,i} \sim Unif[0, \delta]$ for $i = 1, \ldots, N_{ij}$</td>
</tr>
<tr>
<td>2. Generate independent compound Poisson process with jumps:</td>
<td>3. Generate deseas. log-spot price path with jumps:</td>
</tr>
<tr>
<td>3a. Set $X_j \leftarrow X_j + \sum_{i=1}^{N_{ij}} U_{j,i} \delta J_{j,i}$ (MR to MRJ)</td>
<td>3a. Set $X_j \leftarrow X_j + \sum_{i=1}^{N_{ij}} 1_{U_{j,i} \leq \delta} J_{j,i} \delta$ (MR to MRJSV)</td>
</tr>
<tr>
<td>3b. Set $X_j \leftarrow X_j + (1 + k \delta / 2)^{-1} \sum_{i=1}^{N_{ij}} 1_{U_{j,i} \geq \delta} J_{j,i}$ (MRSV to MRJSV)</td>
<td>4. Generate option payoff samples:</td>
</tr>
<tr>
<td>4a. Set $C \leftarrow e^{-rT} \left( \sum_{j=0}^{n} \exp(X_j)/(n+1) - \tilde{K} \right)^+$ (arithmetic average)</td>
<td>4a. Set $C \leftarrow e^{-rT} \left( \exp \left( \sum_{j=0}^{n} X_j/(n+1) \right) - \tilde{K} \right)^+$ (geometric average)</td>
</tr>
<tr>
<td>4b. Set $G \leftarrow e^{-rT} \left( \exp \left( \sum_{j=0}^{n} X_j/(n+1) \right) - \tilde{K} \right)^+$ (control variate)</td>
<td>4c. Set $C_b \leftarrow C - b^*(G - E(G))$</td>
</tr>
<tr>
<td>5. Repeat steps 1–4 for all $M$ simulations</td>
<td>5. Repeat steps 1–4 for all $M$ simulations</td>
</tr>
<tr>
<td>6. <strong>Return</strong> option price estimate $E(C_b) \leftarrow \frac{1}{M} \sum C_b$</td>
<td>6. <strong>Return</strong> option price estimate $E(C_b) \leftarrow \frac{1}{M} \sum C_b$</td>
</tr>
<tr>
<td>7. <strong>Return</strong> standard error $\frac{1}{M-1} \sum (C_b - E(C_b))^2$</td>
<td>7. <strong>Return</strong> standard error $\frac{1}{M-1} \sum (C_b - E(C_b))^2$</td>
</tr>
</tbody>
</table>
Table 4: Arithmetic Asian option prices

This table reports the price estimates of at-the-money (ATM), in-the-money (ITM) and out-of-the-money (OTM) 1-month to maturity arithmetic Asian call options with daily monitoring, obtained by implementing the control variate Monte Carlo simulation approach in Section 5.1 (see also Table 3) with 100,000 simulations, for each model (MR, MRJ, MRSV, MRJSV) and market (CB, GO, CL, HU, HO). The strike price is set equal to 100%, 95% and 105% of the spot price, resp., ATM, ITM and OTM options. The current spot price and variance are set equal to their long-run mean levels. Columns 2–4 report option prices, columns 5–7 option prices relative to the spot price (%), columns 8–9 ITM and OTM option prices relative to the price of the corresponding ATM option (%).

<table>
<thead>
<tr>
<th>Market</th>
<th>Option prices</th>
<th>Relative to spot (%)</th>
<th>Relative to ATM (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>ITM</td>
<td>ATM</td>
<td>OTM</td>
</tr>
<tr>
<td>Brent Crude Oil (CB)</td>
<td>70.32 $/bbl</td>
<td></td>
<td></td>
</tr>
<tr>
<td>MR</td>
<td>2.68</td>
<td>0.84</td>
<td>0.16</td>
</tr>
<tr>
<td>MRJ</td>
<td>3.01</td>
<td>1.02</td>
<td>0.22</td>
</tr>
<tr>
<td>MRSV</td>
<td>3.27</td>
<td>1.48</td>
<td>0.55</td>
</tr>
<tr>
<td>MRJSV</td>
<td>3.51</td>
<td>1.62</td>
<td>0.60</td>
</tr>
<tr>
<td>Gasoil (GO) - spot price 554.80 $/mt</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>MR</td>
<td>20.87</td>
<td>6.62</td>
<td>1.31</td>
</tr>
<tr>
<td>MRJ</td>
<td>22.26</td>
<td>7.41</td>
<td>1.61</td>
</tr>
<tr>
<td>MRSV</td>
<td>27.75</td>
<td>13.82</td>
<td>6.11</td>
</tr>
<tr>
<td>MRJSV</td>
<td>27.54</td>
<td>13.49</td>
<td>5.58</td>
</tr>
<tr>
<td>WTI Crude Oil (CL) - spot price 72.26 $/bbl</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>MR</td>
<td>3.65</td>
<td>1.36</td>
<td>0.33</td>
</tr>
<tr>
<td>MRJ</td>
<td>3.70</td>
<td>1.42</td>
<td>0.38</td>
</tr>
<tr>
<td>MRSV</td>
<td>4.21</td>
<td>2.10</td>
<td>0.89</td>
</tr>
<tr>
<td>MRJSV</td>
<td>4.20</td>
<td>2.05</td>
<td>0.85</td>
</tr>
<tr>
<td>Gasoline (HU) - spot price 1.84 c$/gal</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>MR</td>
<td>5.6</td>
<td>1.9</td>
<td>0.5</td>
</tr>
<tr>
<td>MRJ</td>
<td>6.7</td>
<td>2.6</td>
<td>0.8</td>
</tr>
<tr>
<td>MRSV</td>
<td>10.1</td>
<td>5.8</td>
<td>3.3</td>
</tr>
<tr>
<td>MRJSV</td>
<td>9.7</td>
<td>5.5</td>
<td>3.0</td>
</tr>
<tr>
<td>Heating Oil (HO) - spot price 1.75 c$/gal</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>MR</td>
<td>7.1</td>
<td>2.5</td>
<td>0.6</td>
</tr>
<tr>
<td>MRJ</td>
<td>7.5</td>
<td>2.8</td>
<td>0.8</td>
</tr>
<tr>
<td>MRSV</td>
<td>9.3</td>
<td>4.7</td>
<td>2.1</td>
</tr>
<tr>
<td>MRJSV</td>
<td>9.4</td>
<td>5.0</td>
<td>2.3</td>
</tr>
</tbody>
</table>
Table 5: Hedge performance comparisons

This table reports for each market the mean and standard deviation (std) of the simulated hedging error distribution, in monetary terms (CB - $/bbl, GO - $/mt, CL - $/bbl, HU - c$/gal, HO - c$/gal), for given driving process (MR, MRJ, MRSV, MRJSV) and delta hedge portfolios formed accordingly (Panels A.1, B.1, C.1, D.1), as well as the % increases (positive signs) or decreases (negative signs) in the standard deviation of the hedging error when the hedge portfolios are misspecified, i.e., formed based on alternative models (Panels A.2, B.2, C.2, D.2). Hedging error is defined as the difference between the value of the delta hedge portfolio and the value of a long 1-month to maturity ATM arithmetic Asian option with daily monitoring for a 1-week hedge period.

<table>
<thead>
<tr>
<th></th>
<th>CB</th>
<th>GO</th>
<th>CL</th>
<th>HU</th>
<th>HO</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Panel A: hedges under the MR environment</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(1) MR hedging error mean</td>
<td>-0.228</td>
<td>-1.812</td>
<td>-0.070</td>
<td>-0.010</td>
<td>-0.005</td>
</tr>
<tr>
<td>std</td>
<td>0.557</td>
<td>4.361</td>
<td>0.559</td>
<td>0.019</td>
<td>0.015</td>
</tr>
<tr>
<td>(2) % changes in std of hedging error due to misspecified model hedge</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>MRJ</td>
<td>-4.21</td>
<td>-2.48</td>
<td>-0.20</td>
<td>-10.06</td>
<td>-2.73</td>
</tr>
<tr>
<td>MRSV</td>
<td>-4.07</td>
<td>-2.79</td>
<td>-0.20</td>
<td>-9.81</td>
<td>-3.07</td>
</tr>
<tr>
<td>MRJSV</td>
<td>-3.64</td>
<td>-2.73</td>
<td>-0.04</td>
<td>-8.12</td>
<td>-2.58</td>
</tr>
<tr>
<td><strong>Panel B: hedges under the MRJ environment</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(1) MRJ hedging error mean</td>
<td>-0.165</td>
<td>-1.674</td>
<td>-0.075</td>
<td>-0.008</td>
<td>-0.005</td>
</tr>
<tr>
<td>std</td>
<td>0.568</td>
<td>4.918</td>
<td>0.702</td>
<td>0.021</td>
<td>0.017</td>
</tr>
<tr>
<td>(2) % changes in std of hedging error due to misspecified model hedge</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>MR</td>
<td>4.19</td>
<td>3.04</td>
<td>0.64</td>
<td>13.57</td>
<td>3.30</td>
</tr>
<tr>
<td>MRSV</td>
<td>-0.16</td>
<td>-1.43</td>
<td>-0.01</td>
<td>-5.16</td>
<td>-1.37</td>
</tr>
<tr>
<td>MRJSV</td>
<td>-0.38</td>
<td>-1.45</td>
<td>-0.25</td>
<td>-6.04</td>
<td>-1.39</td>
</tr>
<tr>
<td><strong>Panel C: hedges under the MRSV environment</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(1) MRSV hedging error mean</td>
<td>-0.201</td>
<td>-1.344</td>
<td>-0.023</td>
<td>-0.007</td>
<td>-0.004</td>
</tr>
<tr>
<td>std</td>
<td>0.765</td>
<td>6.928</td>
<td>0.797</td>
<td>0.033</td>
<td>0.022</td>
</tr>
<tr>
<td>(2) % changes in std of hedging error due to misspecified model hedge</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>MR</td>
<td>8.74</td>
<td>12.11</td>
<td>1.38</td>
<td>35.06</td>
<td>12.12</td>
</tr>
<tr>
<td>MRJ</td>
<td>-0.78</td>
<td>5.53</td>
<td>0.03</td>
<td>12.44</td>
<td>5.10</td>
</tr>
<tr>
<td>MRJSV</td>
<td>-3.47</td>
<td>-0.23</td>
<td>-0.66</td>
<td>-3.73</td>
<td>-1.44</td>
</tr>
<tr>
<td><strong>Panel D: hedges under the MRJSV environment</strong></td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(1) MRJSV hedging error mean</td>
<td>-0.145</td>
<td>-1.714</td>
<td>-0.064</td>
<td>-0.007</td>
<td>-0.003</td>
</tr>
<tr>
<td>std</td>
<td>0.742</td>
<td>6.021</td>
<td>0.813</td>
<td>0.028</td>
<td>0.023</td>
</tr>
<tr>
<td>(2) % changes in std of hedging error due to misspecified model hedge</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>MR</td>
<td>12.07</td>
<td>10.42</td>
<td>1.55</td>
<td>35.23</td>
<td>12.85</td>
</tr>
<tr>
<td>MRJ</td>
<td>2.66</td>
<td>4.64</td>
<td>0.51</td>
<td>14.20</td>
<td>6.23</td>
</tr>
<tr>
<td>MRSV</td>
<td>3.42</td>
<td>0.17</td>
<td>0.49</td>
<td>3.15</td>
<td>1.38</td>
</tr>
</tbody>
</table>
Fig. 1: Model-implied distributions of log-returns and hedging errors

Left panel shows the simulated log-return distributions under the assumption of MR (solid line) and MRJSV (dashed line) driving processes for each market (CB, GO, CL, HU, HO); right panel shows the corresponding hedging error distributions.