Attilio Meucci

(Re)Defining and Managing Diversification

STUDY IT:  www.symmys.com (white papers and code)

DO IT:  Advanced Risk and Portfolio Management® Bootcamp
         www.symmys.com/arpm-bootcamp
Standard approach: Modern Portfolio Theory

\[ w^* \equiv \arg\max_{w \in \mathcal{C}} (\mathbb{E}\{R\} - \lambda \sqrt{\mathbb{V}\{R\}}). \]

Optimal Portfolio = optimal mean-variance weights under constraints
Standard approach: Modern Portfolio Theory

\[ R = \sum_{n=1}^{n} w_n R_n. \]

Portfolio Return = weighted average of asset returns

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Optimal Portfolio = optimal mean-variance weights under constraints

\[ \mathbb{E}\{R\} = w' \mu_R \]
\[ \mathbb{V}\{R\} = w' \Sigma_R w \]

\( \bar{n} \times 1 \) vector of expected asset returns
\( \bar{n} \times \bar{n} \) matrix of asset covariances
Trend 1: From asset-based allocation to factor-based allocation

\[ \max \left( \mathbb{E}\{R\} - \lambda \sqrt{\mathbb{V}\{R\}} \right). \]

Optimal Portfolio = optimal mean-variance
Trend 1: From asset-based allocation to factor-based allocation

\[ R = \sum_{k=1}^{k-1} b_k F_k + U, \]
\[ = \sum_{k=0}^{k} b_k F_k, \]
\[ = \sum_{k} b_k F_k, \]

Portfolio Return = Linear factor model
(factors: momentum, value, ..., PCA,... strategies, ...)

\[ \max(\mathbb{E}\{R\} - \lambda \mathbb{V}\{R\}). \]

Optimal Portfolio = optimal mean-variance
Trend 1: From asset-based allocation to factor-based allocation

\[ R = \sum_{k=1}^{\bar{k}} b_k F_k + U, \]
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\[ = \sum_k b_k F_k, \]

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(factors: momentum, value, ..., PCA, ...
strategies, ...)

Optimal Portfolio = optimal mean-variance
exposures under constraints
Trend 1: From asset-based allocation to factor-based allocation

\[ R = \sum_{k=1}^{\bar{k}} b_k F_k + U, \]
\[ = \sum_{k=0}^{\bar{k}} b_k F_k, \]
\[ = \sum_{k} b_k F_k, \]

Portfolio Return = Linear factor model
(factors: momentum, value, ..., PCA, ...
strategies, ...)

Optimal Portfolio = optimal mean-variance
exposures under constraints

\[ b^* = \arg \max_{b \in \mathcal{C}} (\mathbb{E}\{R\} - \lambda \mathbb{V}\{R\}). \]

\[ \mathbb{E}\{R\} = b' \mu_F \]
\[ \mathbb{V}\{R\} = b' \Sigma_F b \]

\( \bar{k} \times 1 \) vector of factor premia

\( \bar{k} \times \bar{k} \) matrix of factor covariances
Trend 2: from mean-variance to risk parity, or diversification management

\[ R = \sum_{n=1}^{\tilde{n}} w_n R_n. \]

Portfolio Return = weighted average of asset returns

\[ m_n \equiv \frac{1}{Sd\{R\}} \frac{\partial Sd\{R\}}{\partial w_n} = \frac{w_n \left[ \Sigma R w \right]_n}{w' \Sigma R w}, \]

“contributions” to risk
Trend 2: from mean-variance to risk parity, or diversification management

Portfolio Return = weighted average of asset returns

\[ R = \sum_{n=1}^{\bar{n}} w_n R_n. \]

Optimal Portfolio = equal “contributions” to risk

\[ m_1 = m_2 = \cdots \]

\[ m_n \equiv \frac{1}{\mathbb{S}d\{R\}} \frac{\partial \mathbb{S}d\{R\}}{\partial w_n} = \frac{w_n \Sigma R w}{w' \Sigma R w}, \]

“contributions” to risk

\[ \bar{n} \times \bar{n} \text{ matrix of asset covariances} \]
Diversification: Effective Number of Bets

Portfolio Return = weighted average of asset returns

\[ R = \sum_{n=1}^{n} w_n R_n. \]
Diversification: Effective Number of Bets

\[ R_w \equiv w' \mathbf{R} \]

\[ \text{Var} \{ R_w \} \equiv \sum_{n=1}^{N} \text{Var} \{ w_n R_n \} \]

if correlations = 0
Diversification: Effective Number of Bets

\[ R_w \equiv w' \mathbf{R} \]

\[ \text{Var} \{ R_w \} \neq \sum_{n=1}^{N} \text{Var} \{ w_n R_n \} \]

if correlations \( \neq 0 \)
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Diversification: Effective Number of Bets

\[ R_w \equiv w' \mathbf{R}. \]

\[ \Sigma \equiv \text{Cov}\{ \mathbf{R} \} \]

\[ '\Sigma \equiv \mathbf{E} \Lambda \mathbf{E}' \]

PCA

\[ \mathbf{E} \equiv (\mathbf{e}_1, \ldots, \mathbf{e}_N) \]

eigenvectors

\[ \Lambda \equiv \text{diag} (\lambda_1^2, \ldots, \lambda_N^2) \]

eigenvalues
Diversification: Effective Number of Bets

\[ R_w \equiv w' R. \]

\[ \Sigma \equiv \text{Cov}\{R\} \]

\[ \Sigma \equiv E \Lambda E' \]

PCA

eigenvectors

uncorrelated, maximum variance portfolios

\[ e_n \equiv \arg\max_{e'e \equiv 1} \{e' \Sigma e\} \]

\[ e' \Sigma e_j \equiv 0, \text{ for all existing } e_j \]

\[ \Lambda \equiv \text{diag}(\lambda_1^2, \ldots, \lambda_N^2) \]

eigenvalues

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\( R_w \equiv w' \Sigma \).

\[ \Sigma \equiv \text{Cov}\{R\} \]

\[ '\Sigma \equiv E \Lambda E' \]

**PCA**

\[ E \equiv (e_1, \ldots, e_N) \]

\[ e_n \equiv \arg \max_{e' \equiv 1} \{e' \Sigma e\} \]

uncorrelated, maximum variance portfolios

\[ e' \Sigma e_j \equiv 0, \text{ for all existing } e_j \]

**eigenvectors**

\[ \Lambda \equiv \text{diag} (\lambda_1^2, \ldots, \lambda_N^2) \]

**eigenvalues**

\[ \lambda_n^2 \equiv \text{Var}\{e'_n R\} \]

variances of uncorrelated, maximum variance portfolios

Diversification: Effective Number of Bets
Diversification: Effective Number of Bets

\[ R_w \equiv w' \Sigma \]

\[ \Sigma \equiv \text{Cov}\{\mathbf{R}\} \]

\[ '\Sigma \equiv \mathbf{E} \Lambda \mathbf{E}' \]

PCA

Eigenvectors

\[ \mathbf{E} \equiv (e_1, \ldots, e_N) \]

\[ e_n \equiv \arg\max_{e' \equiv 1} \{e'\Sigma e\} \]

Principal portfolios

\[ e' \Sigma e_j \equiv 0, \text{ for all existing } e_j \]

Eigenvalues

\[ \Lambda \equiv \text{diag}\left(\lambda_1^2, \ldots, \lambda_N^2\right) \]

\[ \lambda_n^2 \equiv \text{Var}\{e'_n \mathbf{R}\} \]

Principal variances
Diversification: Effective Number of Bets

\[ R_w \equiv w'\mathbf{R}. \]

\[ \Sigma \equiv \text{Cov}\{\mathbf{R}\} \]

\[ '\Sigma \equiv \mathbf{E} \Lambda \mathbf{E}' \]

\[ \mathbf{E} \equiv (e_1, \ldots, e_N) \]

\[ \Lambda \equiv \text{diag}\left(\lambda_1^2, \ldots, \lambda_N^2\right) \]

- principal portfolio 1
- principal portfolio 2
- principal variances
- principal portfolios
- PCA
Diversification: Effective Number of Bets

\[ R_w \equiv w' R. \]

\[ \Sigma \equiv \text{Cov}\{R\} \]

\[ \Sigma \equiv \mathbf{E} \Lambda \mathbf{E}' \]

\[ \tilde{R} \equiv \mathbf{E}^{-1} R \]

return of principal portfolios
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Diversification: Effective Number of Bets

\[ R_w \equiv w' \mathbf{R} \]

\[ \Sigma \equiv \text{Cov}\{\mathbf{R}\} \]

\[ '\Sigma \equiv \mathbf{E} \Lambda \mathbf{E}' \]

\[ \tilde{\mathbf{R}} \equiv \mathbf{E}^{-1} \mathbf{R} \quad \text{return of principal portfolios} \]

\[ \tilde{\mathbf{w}} \equiv \mathbf{E}^{-1} \mathbf{w} \quad \text{weights of original portfolio on principal portfolios} \]
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\[ R_w \equiv w' \tilde{R}. \]

\[ \Sigma \equiv \text{Cov}\{R\} \]

\[ \Sigma \equiv E \Lambda E' \]

\[ \tilde{R} \equiv E^{-1}R \]

\[ \tilde{w} \equiv E^{-1}w. \]

\[ \text{return of principal portfolios} \]

\[ \text{weights of original portfolio on principal portfolios} \]
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**Diversification: Effective Number of Bets**

\[ R_w \equiv w' \hat{R} \]

\[ \text{Var} \{ R_w \} \neq \sum_{n=1}^{N} \text{Var} \{ w_n R_n \} \]

---

**Total Variance**

**Variance Concentration Curve**

\[ \nu_n \equiv \hat{w}_n^2 \lambda_n^2 \]

---

**Return of Principal Portfolios**

\[ \hat{R} \equiv E^{-1} R \]

**Weights of Original Portfolio on Principal Portfolios**

\[ \hat{w} \equiv E^{-1} w \]

**Variance Concentration Curve**

**Contribution to Original Portfolio Variance from n-th Principal Portfolio:**

\[ \text{Var} \{ R_w \} \equiv \sum_{n=1}^{N} \nu_n \]
\[ R_w \equiv w' \hat{\mathbf{R}}. \]

**Volatility Concentration Curve**

- **Return of Principal Portfolios**
  \[ \tilde{\mathbf{R}} \equiv \mathbf{E}^{-1} \mathbf{R} \]

- **Weights of Original Portfolio on Principal Portfolios**
  \[ \tilde{\mathbf{w}} \equiv \mathbf{E}^{-1} \mathbf{w} \]

- **Variance Concentration Curve**
  \[ \nu_n \equiv \tilde{\mathbf{w}}_n^2 \lambda_n^2 \]

- **Volatility Concentration Curve**
  \[ s_n \equiv \frac{\tilde{\mathbf{w}}_n^2 \lambda_n^2}{\text{Sd} \{ R_w \}} \]

**Contribution to Original Portfolio Volatility from n-th Principal Portfolio:** "hot spots"
weights of original portfolio on principal portfolios

\[ \tilde{\mathbf{w}} = E^{-1} \mathbf{w} \]

return of principal portfolios

\[ \tilde{\mathbf{R}} = E^{-1} \mathbf{R} \]

weights of original portfolio on principal portfolios

\[ \mathbf{w}_n \equiv \tilde{\mathbf{w}}^2 \lambda_n \]

variance concentration curve

\[ s_n \equiv \frac{\tilde{\mathbf{w}}^2 \lambda_n^2}{\text{Sd} \{ R_w \}} \]

volatility concentration curve

\[ p_n \equiv \frac{\tilde{\mathbf{w}}^2 \lambda_n^2}{\text{Var} \{ R_w \}} \]

diversification distribution

contribution to original portfolio r-square from n-th principal portfolio

\[ R_w \equiv \tilde{\mathbf{w}}' \tilde{\mathbf{R}}. \]
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Diversification: Effective Number of Bets

\[ R_w \equiv w' \tilde{R}. \]

\[ \tilde{R} \equiv E^{-1} R \]

return of principal portfolios

\[ \tilde{w} \equiv E^{-1} w. \]

weights of original portfolio on principal portfolios

\[
\begin{align*}
\nu_n & \equiv \tilde{w}_n^2 \lambda_n^2 \\

s_n & \equiv \frac{\tilde{w}_n^2 \lambda_n^2}{\text{Sd} \{R_w\}} \\
p_n & \equiv \frac{\tilde{w}_n^2 \lambda_n^2}{\text{Var} \{R_w\}}
\end{align*}
\]

variance concentration curve

\[ \uparrow \]

volatility concentration curve

\[ \uparrow \]

diversification distribution
weights of original portfolio on principal portfolios

\[ \tilde{\mathbf{w}} \equiv \mathbf{E}^{-1} \mathbf{w} \]

weights of original portfolio on principal portfolios

\[ \mathbf{R}_w \equiv \tilde{\mathbf{w}}' \tilde{\mathbf{R}}. \]

return of principal portfolios

\[ \tilde{\mathbf{R}} \equiv \mathbf{E}^{-1} \mathbf{R} \]

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Diversification: Effective Number of Bets

\[ \nu_n \equiv \tilde{w}_n^2 \lambda_n^2 \]

variance concentration curve

\[ s_n \equiv \frac{\tilde{w}_n^2 \lambda_n^2}{\text{Sd}\{R_w\}} \]

volatility concentration curve

\[ \rho_n \equiv \frac{\tilde{w}_n^2 \lambda_n^2}{\text{Var}\{R_w\}} \]

diversification distribution: “probability mass”
Diversification: Effective Number of Bets

\[ N_{\text{Ent}} \equiv \exp \left( - \sum_{n=1}^{N} p_n \ln p_n \right) \]

\[ \Downarrow \]

diversification

\[ \tilde{R} \equiv E^{-1}R \]

return of principal portfolios

\[ \tilde{w} \equiv E^{-1}w \]

weights of original portfolio on principal portfolios

\[
\begin{align*}
\nu_n & \equiv \tilde{w}_n^2 \lambda_n^2 \\
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p_n & \equiv \frac{\tilde{w}_n^2 \lambda_n^2}{\text{Var} \{ R_w \}}
\end{align*}
\]

variance concentration curve

\[ \Downarrow \]

volatility concentration curve

\[ \Downarrow \]

diversification distribution: “probability mass”
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**Diversification: Effective Number of Bets**

**effective number of bets**

\[ N_{Ent} \equiv \exp \left( - \sum_{n=1}^{N} p_n \ln p_n \right) \]

full concentration \( N_{Ent} \approx 1 \)

weights

\[ p_n \equiv \frac{\tilde{w}_n^2 \lambda_n^2}{\text{Var} \{ R_w \}} \]

diversification distribution: “probability mass”
effective number of bets

\[ N_{\text{Ent}} \equiv \exp \left( - \sum_{n=1}^{N} p_n \ln p_n \right) \]

full concentration \[ N_{\text{Ent}} \approx 1 \]

full diversification \[ N_{\text{Ent}} \approx N \]

weights

diversification distribution

\[ p_n \equiv \frac{\tilde{\omega}_n^2 \lambda_n^2}{\text{Var} \{ R_w \}} \]

diversification distribution: “probability mass”
full concentration $N_{\text{Ent}} \approx 1$

full diversification $N_{\text{Ent}} \approx N$

effective number of bets

$$N_{\text{Ent}} \equiv \exp \left( - \sum_{n=1}^{N} p_n \ln p_n \right)$$

Diversification management

mean-diversification frontier

$$w_{\varphi} \equiv \arg\max_{w \in C} \{ \varphi \mu'w + (1 - \varphi) N_{\text{Ent}}(w) \}$$
Next Steps: Minimal Torsion Bets

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Assets

Modern Portf. Theory

Factors

Factor-Based Allocation
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Next Steps: Minimal Torsion Bets

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## Next Steps: Minimal Torsion Bets

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i) Factors on Demand: \( R = \sum_{k=1}^{k} b_k F_k + U \), best portfolio-specific linear factor model
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#### Next Steps: Minimal Torsion Bets

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2) Effective Num. Bets | Factor-Based Allocation |

#### Original Factors

- Minimum-Torsion Bets
- Principal Components Bets

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### i) Factors on Demand

\[
R = \sum_{k=1}^{k} b_k F_k + U,
\]

*best* portfolio-specific linear factor model

### ii) Minimal Torsion Bets

\[
\hat{F}_{MT} \equiv \hat{t}_{MT} F
\]

*uncorrelated* factors closest to factors

\[
\text{argmin}_{C_r(tF)=i_{\tilde{n} \times \tilde{n}}} NTE\{tF, F\}
\]

\[
\sqrt{\sum_{\tilde{n}} \forall \{Z_{\tilde{n}} - (F_{\tilde{n}} / Sd\{F_{\tilde{n}}\})\}}
\]
### Next Steps: Minimal Torsion Bets

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**i) Factors on Demand**

\[ R = \sum_{k=1}^{k} b_k F_k + U, \text{ best portfolio-specific linear factor model} \]

**ii) Minimal Torsion Bets:**

\[ \tilde{F}_{MT} \equiv \hat{t}_{MT} F \quad \text{uncorrelated factors closest to factors} \]

\[
\text{argmin}_{Cr\{tF\} = i_{\bar{n} \times \bar{n}}} NTE\{tF, F\} \approx \frac{dg(\sigma_F) c^{-1}}{dg(\sigma_F)^{-1}}
\]

\[
\sqrt{\sum_{n} \frac{\sqrt{\sum_{n} Z_n - (F_n / Sd\{F_n\})}}{\sqrt{\sum_{n} Y\{Z_n - (F_n / Sd\{F_n\})\}}}}
\]

Riccati root of correlation
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Next Steps: Minimal Torsion Bets

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i) Factors on Demand

\[ R = \sum_{k=1}^{k} b_F F_k + U, \]  
best portfolio-specific linear factor model

ii) Minimal Torsion Bets:

\[ \hat{F}_{MT} = \hat{t}_{MT} F \]  
uncorrelated factors closest to factors

\[
\arg\min_{C(r\{tF\})=i_{\hat{n}x\hat{n}}} NTE\{tF, F\} \approx \frac{dg(\sigma_F) c^{-1}}{dg(\sigma_F)^{-1}}
\]
Riccati root of correlation

ii) Diversification Distribution and Effective Number of Minimum Torsion Bets

\[
p_{MT}(b) = \left( \hat{t}_{MT}^{-1} b \right) \circ \left( \hat{t}_{MT} \Sigma_F b \right) \quad \Rightarrow \quad N_{MT}(b) = e^{-p_{MT}(b)'} \ln p_{MT}(b)
\]
Effective Number of Bets (normalized)

Weights / Probabilities

Minimum-Torsion Diversification Distribution

Portfolio Weights

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The Effective Number of PCA Bets in the S&P500 is close to 1, since the first PCA factor loadings are similar to the weights of the stocks in the S&P500.

The Effective Number of Minimal Torsion Bets in the S&P500 yields intuitive results.
Marginal Contributions

Portfolio Weights

Weights / Probabilities

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Stocks

Time

Sep-12 May-10 Jan-08 Sep-05 May-03 Jan-01
Next Steps: Minimal Torsion Bets

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<tr>
<td>Risk contrib.</td>
<td>Marginal Contributions</td>
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<tr>
<td>Expression</td>
<td>$m \approx \frac{b \circ (\Sigma_F b)}{b' \Sigma_F b}$</td>
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<tr>
<td>Meaning</td>
<td>spurious contributions from original factors</td>
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<tr>
<td>Properties</td>
<td>$\sum_n m_n = 1, \ m_n \leq 0$</td>
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Effective Number of Bets: http://symmys.com/node/199

Factors on Demand: http://symmys.com/node/164

Minimal Torsion Bets: http://symmys.com/node/599